Integration

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1 Preface

This paper gives a short overview of a new integration technique, using difference inequalities instead of limits or infinite sums. The goal of this paper is to show that integration and differentiation can be introduced with this method.

Content

2 Introduction

Integration means to determine the 'area' beneath a function. We will introduce a new technique to determine this area, without the usual analytical methods introduced to overcome some dubious notions like differentials etc. Our method does not require the usual notions of limits or infinite sums, so that the traditional analytical concepts are not required by this concept.

Our method states a system of difference inequalities and shows that the usual well known properties of integral functions can be derived from this, including concepts of differentiation as well as the usual rules for integration and differentiation. The aim of this paper is therefore to state the corresponding system of difference inequalities, where the integral function I_f is expressed as a difference, bounded by expressions with the function f. We start with some simple applications of this concept, deriving the fundamental integral functions of polynomials, exponential functions, logarithms and trigonometric functions.

Then we show the properties of this integral function, particularly its uniqueness, but also its meaning as the area function. Also the well known properties continuity, linearity, additivity, and translocitiy.

The inversion of integration is differentiation that is shown in the same way by a similar system of difference inequalities. Also some of standard derivatives are derived, as well as the standard interpretation of derivatives.

The next chapter derives some of the well known concepts of rules for differentiation and integration by use of our definition. Again none of the concepts of standard analysis are required to achieve this goal. Finally we consider some miscellaneous concepts.

3 The fundamental inequalities

The reader is assumed to be known with notions from algebra, i.e. inequalities, real or rational numbers as well as functions and continuos functions.

3.1 The integral of a function

Let f be a function. Let f be monotonously ascending in interval $[u,v]$ and let for all x and Δx, where

$$u \leq x < x + \Delta x \leq v$$

hold the following inequalities

$$f(x) \cdot \Delta x \leq I_f(x+\Delta x) - I_f(x) \leq f(x+\Delta x) \cdot \Delta x . \qquad (1)$$

Then I_f is the *integral function* of f with the properties

1. I_f is uniquely defined by inequalities (1), besides a constant addend.
2. $I_f(x)$–$I_f(u)$ is the areal function beneath the function f, if $f(x)$ is positive in that interval $[u,v]$.
3. I_f is continuos.

We will prove the second property in section 3.3. Formally, (1) is a *system of difference inequalities*, since there are <u>two</u> inequalities, while I_f is defined by its <u>difference</u>, only. We call this system of difference inequalities from now on 'the inequalities'.

If f is monotonously descending in that interval, we get an analogous system of difference inequalities:

$$f(x) \cdot \Delta x \geq I_f(x+\Delta x) - I_f(x) \geq f(x+\Delta x) \cdot \Delta x , \qquad (2)$$

with exactly the same properties as that in (1).

3.2 Some examples of integral functions

We give here some simple examples of integral functions.

3.2.1 Integral function of a constant *a*

Let $f(x) = a$ be a constant, then $I_f(x) = a \cdot x$, since

$$a \cdot \Delta x \leq a \cdot (x+\Delta x) - a \cdot x = a \cdot \Delta x \leq a \cdot \Delta x .$$

3.2.2 Integral function of *x*

Let $f(x) = x$, then $I_f(x) = ½ \cdot x^2$, since

$$x\cdot\Delta x\le 1/2\cdot(x+\Delta x)^2-1/2\cdot x^2=x\cdot\Delta x+1/2\cdot\Delta^2 x\le(x+\Delta x)\cdot\Delta x\,.$$

3.2.3 Integral function of x^2

Let $f(x) = x^2$, then $I_f(x) = 1/3\cdot x^3$, since for positive x holds

$$x^2\cdot\Delta x\le 1/3\cdot(x+\Delta x)^3-1/3\cdot x^3=x^2\cdot\Delta x+x\cdot\Delta^2 x+1/3\cdot\Delta^3 x\le(x+\Delta x)^2\cdot\Delta x\,.$$

3.2.4 Integral function of x^3

Let $f(x) = x^3$, then $I_f(x) = ¼\cdot x^4$, since for all x holds

$$x^3\cdot\Delta x\le 1/4\cdot(x+\Delta x)^4-1/4\cdot x^4=x^3\cdot\Delta x+1.5\cdot x^2\cdot\Delta^2 x+x\cdot\Delta^3 x+1/4\cdot\Delta^4 x\le(x+\Delta x)^3\cdot\Delta x\,.$$

3.2.5 Integral function of e^x

Let $f(x) = e^x$, then $I_f(x) = e^x$, since for this function holds for all z that $1+z < e^z$, and thus with $z = –\Delta x$

$$1+\Delta x\le e^{\Delta x}\Rightarrow e^x\cdot\Delta x\le e^{x+\Delta x}-e^x,$$

and with $z = –\Delta x$ follows

$$1-\Delta x\le e^{-\Delta x}\Rightarrow e^x-e^{x-\Delta x}\le e^x\cdot\Delta x\Rightarrow e^{x+\Delta x}-e^x\le e^{x+\Delta x}\cdot\Delta x\,.$$

3.2.6 Integral function of ln x

Let be $f(x) = \ln x$, then for positive x follows $I_f(x) = x\cdot(\ln x–1)$. We have to show

$$\Delta x\cdot\ln x\le(x+\Delta x)\cdot(\ln(x+\Delta x)-1)-x\cdot(\ln x-1)\le\Delta x\cdot\ln(x+\Delta x)\,. \tag{3}$$

For each positive z holds: $\ln z \le z–1$. With $z = x/(x+\Delta x)$ follows

$$\ln\frac{x}{x+\Delta x}=\ln x-\ln(x+\Delta x)\le\frac{x}{x+\Delta x}-1=\frac{-\Delta x}{x+\Delta x},$$

or after reordering and multiplication by the numerator

$$0\le(x+\Delta x)\cdot\ln(x+\Delta x)-(x+\Delta x)\cdot\ln x-\Delta x=(x+\Delta x)\cdot\ln(x+\Delta x)-x\cdot\ln x-\Delta x\cdot\ln x-\Delta x\,.$$

Thus follows the left inequality of (1), after reordering and adding $x–x$

$$\Delta x\cdot\ln x\le(x+\Delta x)\cdot\ln(x+\Delta x)-x-\Delta x-x\cdot\ln x+x=(x+\Delta x)\cdot(\ln(x+\Delta x)-1)-x\cdot(\ln x-1)\,.$$

Since $z–1 \ge \ln z$ for each positive z, follows with $z = 1+\Delta x/x$

$$\frac{\Delta x}{x}\ge\ln(1+\frac{\Delta x}{x})=\ln(x+\Delta x)-\ln x\,.$$

Multiplying with positive x and inserting some further terms yields

$$\Delta x\cdot\ln(x+\Delta x)\ge x\cdot\ln(x+\Delta x)+\Delta x\cdot\ln(x+\Delta x)-x-\Delta x-x\cdot\ln x+x\,.$$

After reordering and further manipulations follows the right inequality of (1).

3.2.7 Integral function of sin x

Let be $f(x) = \sin x$, then follows $I_{\sin}(x) = –\cos x$. For the left inequality of (1) we have to show

$$\Delta x \cdot \sin x \leq -\cos(x+\Delta x)+\cos x = \cos x - \cos x \cdot \cos \Delta x + \sin x \cdot \sin \Delta x\,,$$

provided we consider only the interval [0,π/2]. Since sin x is positive follows

$$\Delta x \leq \cot x - \cot x \cdot \cos \Delta x + \sin \Delta x = \cot x \cdot (1-\cos \Delta x) + \sin \Delta x\,. \tag{4}$$

With $x = \pi/2 - \Delta x$ follows because of $\cot(\pi/2-\Delta x) = \tan \Delta x$

$$\Delta x \leq \cot(\pi/2-\Delta x)\cdot(1-\cos \Delta x)+\sin \Delta x = \tan \Delta x - \sin \Delta x + \sin \Delta x = \tan \Delta x\,,$$

which holds for each positive $\Delta x < \pi/2$. If $x < \pi/2-\Delta x$, then its cotangens is larger than for $\pi/2-\Delta x$, and the right hand side of (4) is larger as well.

For the right hand side of (1) we have to show

$$\begin{aligned}\cos x - \cos(x+\Delta x) &= \cos((x+\Delta x)-\Delta x) - \cos(x+\Delta x) = \\ &\cos((x+\Delta x)-\Delta x) - \cos(x+\Delta x) = \\ &\cos(x+\Delta x)\cdot\cos -\Delta x - \sin(x+\Delta x)\cdot \sin -\Delta x - \cos(x+\Delta x) \leq \\ &\leq \Delta x \cdot \sin(x+\Delta x)\,,\end{aligned}$$

or

$$\cos(x+\Delta x)\cdot(\cos \Delta x - 1) \leq (\Delta x - \sin \Delta x)\cdot \sin(x+\Delta x)\,.$$

The left hand side is always negative, the right positive.

3.3 The areal property

We give here an interpretation of the integral function, which proves also its uniqueness. Let the monotonously ascending function f be positive on interval $[u,v]$. Then holds (1) for its integral function I_f on each partial interval $[x,x+\Delta x]$ from $[u,v]$. The meaning of those inequalities (1) is of course that the product $f(x)\cdot\Delta x$, which is the area of the rectangular strip below the function f in interval $[x,x+\Delta x]$, is less than $I_f(x+\Delta x)-I_f(x)$, while the product $f(x+\Delta x)\cdot\Delta x$, which is the area of the rectangular strip enclosing the area beneath the function f in interval $[x,x+\Delta x]$, is larger than $I_f(x+\Delta x)-I_f(x)$. Thus the system of difference inequalities (1) describes a lower and upper bound of an areal function, where the lower bound – the set of rectangular strips below the curve – is an area that lies completely inside the area beneath the function, and the upper bound encloses the area beneath the function completely.

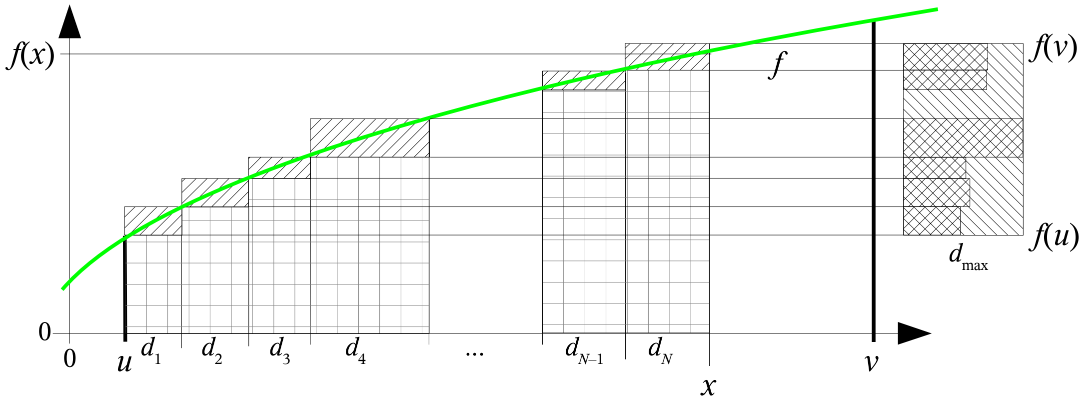
For any partitioning $u=x_0<x_1<x_2<..<x_N=x$ of that interval $[u,x]$ we have for any partial interval $[x_i,x_{i+1}]$ inequalities (1), so that summing of all those inequalities we get the sum of areas of rectangular strips below that function f, (that lies completely inside the area beneath the function f) and the sum of areas of rectangular strips enclosing the area beneath function f.

The areal functions $I_f(x_{i+1})-I_f(x_i)$ as difference of the integral functions I_f sum to

$$\sum_{i=0}^{N-1} I_f(x_{i+1}) - I_f(x_i) = I_f(x) - I_f(u)\,.$$

This shows that the difference of the integral functions $I_f(x)-I_f(u)$ is bounded by the sum of areas of rectangular strips inside the area beneath function f, and the sum of areas of rectangular strips enclosing the area beneath function f. This is completely independent of the particular partitioning $u=x_0<x_1<x_2<..<x_N=x$. Also it can easily be shown that the difference between those bounding areas is obviously bounded by $(v-u)\cdot d_{\max}$, where $d_{\max}$ is the maximum width of all of those strips, or the maximum of all $x_{i+1}-x_i$. Since this width $d_{\max}$ can be assumed to be arbitrarily small, the difference $I_f(x)-I_f(u)$ is uniquely defined and its meaning is the area beneath the function f, provided f is posi-

tive. Otherwise the area might become negative or zero for non zero functions, which requires a particular interpretation.



To make this result quite clear, the outcome of these considerations shows that the area of any arbitrary rectangular partitioning of the area beneath the function f, and any arbitrary rectangular partitioning that encloses the area beneath the function f, is a lower and upper bound of the value of the areal function $I_f(x)–I_f(u)$.

3.4 Properties of the integral function

We sketch here some properties of integral function as defined in (1) or (2).

3.4.1 Continuity of integral function

From definition (1) or (2) of integral function follows immediately its continuity, since in case (1) with the finite difference $a = f(x+\Delta x)–f(x)$ there is for any given δ a $\Delta x < \delta/a$, so that

$$0 \le I_f(x+\Delta x) - I_f(x) \le a \cdot \Delta x < \delta .$$

3.4.2 Linearity of integral function

For each integral function we can prove its linearity: $I_{a \cdot f} = a \cdot I_f$.

$$a \cdot f(x) \cdot \Delta x \le a \cdot I_f(x+\Delta x) - a \cdot I_f(x) \le a \cdot f(x+\Delta x) \cdot \Delta x .$$

Analogous for (2). this follows from multiplication with a. If $a < 0$ an ascending function becomes descending, and vice versa, and the inequalities signs have to be altered.

3.4.3 Additivity of integral function

Additivity: $I_{f+g} = I_f + I_g$

$$(f(x)+g(x)) \cdot \Delta x \le I_f(x+\Delta x) - I_f(x) + I_g(x+\Delta x) - I_g(x) \le (f(x+\Delta x)+g(x+\Delta x)) \cdot \Delta x .$$

Follows from addition of the inequalities, when both equation are monotonously ascending or descending. We also write $I_{f+g} = I_f + I_g$:

$$(f(x)+g(x)) \cdot \Delta x \le I_{f+g}(x+\Delta x) - I_{f+g}(x) \le (f(x+\Delta x)+g(x+\Delta x)) \cdot \Delta x .$$

If not both functions are ascending or descending, some further properties are to be considered.

3.4.4 Translocation invariance of integral function

Translocation invariance: $I_{f(x+c)}(x+c) = I_f(x)$.

$$f(x+c)\cdot\Delta x \le I_f(x+c+\Delta x) - I_f(x+c) \le f(x+c+\Delta x)\cdot\Delta x .$$

This follows since shifting in x-direction concerns both functions, f and I_f; formally it can be shown by substitution of $z = x+c$; and since $y+c = x+c+\Delta x$ is also $\Delta z = y+c - (x+c) = y-x = \Delta x$, thus

$$f(z)\cdot\Delta z \le I_f(z+\Delta z) - I_f(z) \le f(z+\Delta z)\cdot\Delta z .$$

For positive c both functions are shifted to the left, for negative c to the right.

3.4.5 Integral function of the polynomial function

From the formulae

$$\begin{aligned} x^k\cdot\Delta x &\le \frac{1}{k+1}\cdot\sum_{i=0}^{k+1}\binom{k+1}{i}\cdot x^{k+1-i}\cdot\Delta^i x - \frac{1}{k+1}\cdot x^{k+1} = \\ &= \frac{1}{k+1}\cdot\sum_{i=1}^{k+1}\binom{k+1}{i}\cdot x^{k+1-i}\cdot\Delta^i x = \\ &= \sum_{i=1}^{k+1}\binom{k}{i-1}\cdot\frac{1}{i}\cdot x^{k+1-i}\cdot\Delta^i x = \\ &= \sum_{i=0}^{k}\binom{k}{i}\cdot\frac{1}{i+1}\cdot x^{k-i}\cdot\Delta^{i+1} x \le \\ &\le \sum_{i=0}^{k}\binom{k}{i}\cdot x^{k-i}\cdot\Delta^i x\cdot\Delta x = (x+\Delta x)^k\cdot\Delta x . \end{aligned}$$

and linearity and additivity follow the integral function for polynomials

$$I_{\sum a_k\cdot x^k} = \sum \frac{a_k}{k+1}\cdot x^{k+1} .$$

4 The derivative of a function

The inverse function of the integral function is called *derivative function*. It is defined completely analogous to the integral function; however, there are some additional topics to be considered, so that the definition becomes more complicated.

4.1 Definition of *derivative function*

Let f be a continuos function on interval $[u,v]$ and for all x and positive Δx from $[u,v]$ hold the inequalities

$$\begin{aligned} &D_f(x)\cdot\Delta x \le f(x+\Delta x)-f(x) \le D_f(x+\Delta x)\cdot\Delta x, \text{ for } u\le x< x+\Delta x\le v, \\ &D_f(x-\Delta x)\cdot\Delta x \le f(x)-f(x-\Delta x) \le D_f(x)\cdot\Delta x, \text{ for } u\le x-\Delta x< x\le v, \end{aligned} \tag{5}$$

for a continuos, monotonously *ascending* function D_f, or instead hold the inequalities

$$\begin{aligned} &D_f(x)\cdot\Delta x \ge f(x+\Delta x)-f(x) \ge D_f(x+\Delta x)\cdot\Delta x, \text{ for } u\le x< x+\Delta x\le v, \\ &D_f(x-\Delta x)\cdot\Delta x \ge f(x)-f(x-\Delta x) \ge D_f(x)\cdot\Delta x, \text{ for } u\le x-\Delta x< x\le v, \end{aligned} \tag{6}$$

for a continuos, monotonously *descending* function D_f, then the function f is called *continuously differentiable* in interval $[u,v]$ and D_f is called *derivative function* of f. Estimation of D_f is called *differentiation*.

D_f is uniquely defined, is continuos by definition, and the meaning of $D_f(x)$ is the ascent of the tangent of the continuos function f at position x, which is shown at section 4.3.

4.2 Some examples of derivative functions

We give here some simple examples of derivative functions.

4.2.1 Derivative function of a constant *a*

Let $f(x) = a$ be a constant, then $D_f(x) = 0$, since

$$0\cdot\Delta x \le a-a=0 \le 0\cdot\Delta x.$$

4.2.2 Derivative function of *x*

Let $f(x) = x$, then $D_f(x) = 1$, since

$$\begin{aligned} &1\cdot\Delta x \le (x+\Delta x)-x=\Delta x \le 1\cdot\Delta x, \\ &1\cdot\Delta x \le x-(x-\Delta x)=\Delta x \le 1\cdot\Delta x. \end{aligned}$$

4.2.3 Derivative function of x^2

Let $f(x) = x^2$, then $D_f(x) = 2 \cdot x$, since for all x holds

$$2\cdot x\cdot\Delta x\le(x+\Delta x)^2-x^2=2\cdot x\cdot\Delta x+\Delta^2 x\le 2\cdot(x+\Delta x)\cdot\Delta x=2\cdot x\cdot\Delta x+2\cdot\Delta^2 x\,.$$
$$2\cdot(x-\Delta x)\cdot\Delta x=2\cdot x\cdot\Delta x-2\cdot\Delta^2 x\le x^2-(x-\Delta x)^2=2\cdot x\cdot\Delta x-\Delta^2 x\le 2\cdot x\cdot\Delta x\,.$$

4.2.4 Derivative function of x^3

Let $f(x) = x^3$, then $D_f(x) = 3 \cdot x^2$, since for positive interval [0,...], where $x \ge 0$, holds

$$3\cdot x^2\cdot\Delta x\le(x+\Delta x)^3-x^3=3\cdot x^2\cdot\Delta x+3\cdot x\cdot\Delta^2 x+\Delta^3 x\le$$
$$\le 3\cdot(x+\Delta x)^2\cdot\Delta x=3\cdot x^2\cdot\Delta x+6\cdot x\cdot\Delta^2 x+3\cdot\Delta^3 x, \text{ since } x\ge 0,\ \Delta x\ge 0;$$
$$3\cdot(x-\Delta x)^2\cdot\Delta x=3\cdot x^2\cdot\Delta x-6\cdot x\cdot\Delta^2 x+3\cdot\Delta^3 x\le$$
$$\le x^3-(x-\Delta x)^3=3\cdot x^2\cdot\Delta x-3\cdot x\cdot\Delta^2 x+\Delta^3 x\le 3\cdot x^2\cdot\Delta x, \text{ since } 0\le\Delta x\le x\,.$$

For negative intervals [...,0], i.e. $x \le 0$ holds

$$3\cdot x^2\cdot\Delta x\ge(x+\Delta x)^3-x^3=3\cdot x^2\cdot\Delta x+3\cdot x\cdot\Delta^2 x+\Delta^3 x\ge$$
$$\ge 3\cdot(x+\Delta x)^2\cdot\Delta x=3\cdot x^2\cdot\Delta x+6\cdot x\cdot\Delta^2 x+3\cdot\Delta^3 x, \text{ since } x\le -\Delta x,\ \Delta x\ge 0;$$
$$3\cdot(x-\Delta x)^2\cdot\Delta x=3\cdot x^2\cdot\Delta x-6\cdot x\cdot\Delta^2 x+3\cdot\Delta^3 x\ge$$
$$\ge x^3-(x-\Delta x)^3=3\cdot x^2\cdot\Delta x-3\cdot x\cdot\Delta^2 x+\Delta^3 x\ge 3\cdot x^2\cdot\Delta x, \text{ since } x\le 0\le\Delta x\,.$$

4.2.5 Derivative function of e^x

Let $f(x) = e^x$, then $D_f(x) = e^x$, since from section 3.2.5 follows

$$e^x\cdot\Delta x\le e^{x+\Delta x}-e^x\le e^{x+\Delta x}\cdot\Delta x$$
$$e^{(x+\Delta x)-\Delta x}\cdot\Delta x\le e^{x+\Delta x}-e^{(x+\Delta x)-\Delta x}\le e^{x+\Delta x}\cdot\Delta x\,.$$

4.2.6 Derivative function of ln x

Let be $f(x) = \ln x$, then for positive x follows $D_{\ln}(x) = 1/x$. Since $1/x$ is ascending monotonously, we have to show

$$\frac{\Delta x}{x}\ge\ln(x+\Delta x)-\ln x\ge\frac{\Delta x}{x+\Delta x}\,.$$

From (4) follows

$$(x+\Delta x)\cdot(\ln(x+\Delta x)-1)-x\cdot(\ln x-1)=$$
$$x\cdot\ln(x+\Delta x)+\Delta x\cdot\ln(x+\Delta x)-\Delta x-x\cdot\ln x\le\Delta x\cdot\ln(x+\Delta x)$$

or

$$x\cdot\ln(x+\Delta x)-x\cdot\ln x\le\Delta x$$

from which follows the left inequality. Also from (4) follows

$$\Delta x\cdot\ln x\le(x+\Delta x)\cdot(\ln(x+\Delta x)-1)-x\cdot(\ln x-1)=(x+\Delta x)\cdot\ln(x+\Delta x)-\Delta x-x\cdot\ln x$$

or

$$\Delta x\le(x+\Delta x)\cdot\ln(x+\Delta x)-x\cdot\ln x-\Delta x\cdot\ln x=(x+\Delta x)\cdot\ln(x+\Delta x)-(x+\Delta x)\cdot\ln x,$$

what proves the right inequality.

4.2.7 Derivative function of sin *x*

Let be $f(x) = \sin x$, then follows $D_{\sin}(x) = \cos x$. We have to show for interval [0,π/2]

$$\Delta x\cdot\cos x\geq\sin(x+\Delta x)-\sin x\geq\cos(x+\Delta x)\cdot\Delta x.$$

For the left inequality we have

$$\Delta x\cdot\cos x\geq\sin(x+\Delta x)-\sin x=\sin x\cdot\cos\Delta x+\sin\Delta x\cdot\cos x-\sin x$$

or

$$(\Delta x-\sin\Delta x)\cdot\cos x\geq\sin x\cdot(\cos\Delta x-1).$$

where the left hand side is always positive, the right negative. For the other inequality we have

$$\begin{aligned}\sin(x+\Delta x)-\sin x&=\sin(x+\Delta x)-\sin((x+\Delta x)-\Delta x)=\\&=\sin(x+\Delta x)-\sin(x+\Delta x)\cdot\cos(-\Delta x)-\sin(-\Delta x)\cdot\cos(x+\Delta x)\geq\Delta x\cdot\cos(x+\Delta x)\end{aligned}$$

or

$$\tan(x+\Delta x)-\tan(x+\Delta x)\cdot\cos(\Delta x)+\sin(\Delta x)=\sin(\Delta x)+\tan(x+\Delta x)\cdot(1-\cos(\Delta x))\geq\Delta x,$$

which holds for $x = 0$, since then $\tan\Delta x \geq \Delta x$, and for x > 0 the left hand side becomes even larger.

4.3 Interpretation of the derivative function

The derivative function can be interpreted as the tangent of the ascent angle of the function *f*. From (5) follows after dividing by the always positive Δx:

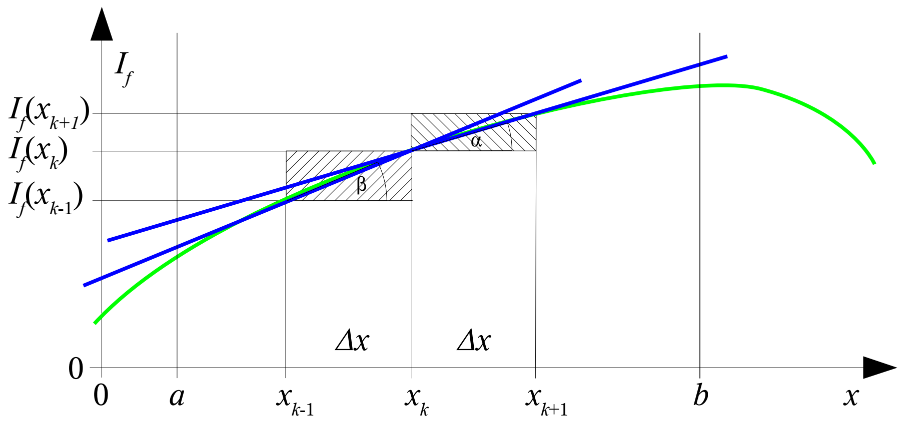
$$D_f(x)\leq\frac{f(x+\Delta x)-f(x)}{\Delta x}\leq D_f(x+\Delta x),$$
$$D_f(x-\Delta x)\leq\frac{f(x)-f(x-\Delta x)}{\Delta x}\leq D_f(x).$$

Since D_f is continuous from definition, its value is uniquely determined by the quotients

$$\frac{f(x+\Delta x)-f(x)}{\Delta x}\text{ or }\frac{f(x)-f(x-\Delta x)}{\Delta x},$$

which must yield the same result, since D_f is continuos. These quotients are usually called *differential quotients*.

Thus there is also a geometric interpretation of the derivative function, namely that $D_f(x)$ is the tangent of the ascent angle of the function *f* at *x*. This shows also, why we need two sets of inequalities to define the derivative function, since there might be a 'kink' in the curve of the function *f*, where the derivatives from left and right side are different. Since our definition requires that both ascent angles are the same, namely $D_f(x)$, for all points *x* of the curve, this cannot occur. Functions *f* that have a derivative function D_f are usually called *differentiable*.



4.4 Properties of derivatives

The derivation D_f of a function f is unique, linear, additive, and translocation invariant.

4.4.1 Construction of derivatives

To construct D_f for a function f, the differential equation can be evaluated. For the exponential function we can write

$$D_{x^k}(x) \le \frac{(x+\Delta x)^k - x^k}{\Delta x} = k\cdot x^{k-1} + \Delta x \cdot (\ldots) \le D_{x^k}(x+\Delta x),$$

where all residual terms hold factor Δx, thus disappear. For the derivative of sine we get

$$D_{\sin}(x) \le \frac{\sin(x+\Delta x) - \sin x}{\Delta x} = \frac{\sin x \cdot (\cos \Delta x - 1) + \cos x \cdot \sin \Delta x}{\Delta x} \le D_{\sin}(x+\Delta x).$$

with more complex considerations. These construction principles can be used to get an idea how D_f might look like. However, the final proof can be stated only by solving the differential equation.

4.4.2 Derivative function of polynomial function

From the formulae

$$\begin{aligned} k\cdot x^{k-1}\cdot\Delta x &\le \sum_{i=0}^{k}\binom{k}{i}\cdot x^{k-i}\cdot\Delta^i x - x^k = \\ &= \sum_{i=1}^{k}\binom{k}{i}\cdot x^{k-i}\cdot\Delta^i x = \\ &= k\cdot\sum_{i=0}^{k-1}\binom{k-1}{i}\cdot\frac{1}{i+1}\cdot x^{k-1-i}\cdot\Delta^i x\cdot\Delta x \le \\ &\le k\cdot\sum_{i=0}^{k-1}\binom{k-1}{i}\cdot x^{k-1-i}\cdot\Delta^i x\cdot\Delta x = k\cdot(x+\Delta x)^{k-1}\cdot\Delta x \end{aligned}$$

and from linearity and additivity follow the formula for the derivative of the polynomial function

$$D_{\sum a_k\cdot x^k} = \sum a_k\cdot k\cdot x^{k-1}.$$

5 Rules for Integration and Differentiation

There are some rules to construct new integral functions and derivative functions from known integral and derivative functions. For example we have already shown additivity

$I_{f+g} = I_f + I_g$,

$D_{f+g} = D_f + D_g$;

or linearity

$I_{a\cdot f} = a\cdot I_f$.

$D_{a\cdot f} = a\cdot D_f$.

Similar rules exist for product and quotient of two functions, particularly for derivativ functions. These rules are now to be derived.

5.1 Product rules

5.1.1 Product rule for Differentiation

Let f and g be monotonously ascending function

$f(x) \leq f(x+\Delta x)$,

$g(x) \leq g(x+\Delta x)$,

what is true for f or $-f$ in any interval. Then

$$0 \leq (f(x+\Delta x)-f(x))\cdot(g(x+\Delta x)-g(x)),$$

or

$$0 \leq f(x+\Delta x)\cdot g(x+\Delta x)-f(x+\Delta x)\cdot g(x)-f(x)\cdot g(x+\Delta x)+f(x)\cdot g(x), \tag{7}$$

or after reordering and addition of $-f(x)\cdot g(x)$ on both sides

$$f(x)\cdot(g(x+\Delta x)-g(x))+g(x)\cdot(f(x+\Delta x)-f(x)) \leq f(x+\Delta x)\cdot g(x+\Delta x)-f(x)\cdot g(x).$$

In (5) we have defined

$$\begin{aligned} D_f(x)\cdot\Delta x &\leq f(x+\Delta x)-f(x) \leq D_f(x+\Delta x)\cdot\Delta x, \\ D_g(x)\cdot\Delta x &\leq g(x+\Delta x)-g(x) \leq D_g(x+\Delta x)\cdot\Delta x. \end{aligned} \tag{8}$$

Then follows

$$\begin{aligned} (f(x)\cdot D_g(x)+g(x)\cdot D_f(x))\cdot\Delta x &= f(x)\cdot D_g(x)\cdot\Delta x+g(x)\cdot D_f(x)\cdot\Delta x \leq \\ &\leq f(x)\cdot(g(x+\Delta x)-g(x))+g(x)\cdot(f(x+\Delta x)-f(x)) \leq \\ &\leq f(x+\Delta x)\cdot g(x+\Delta x)-f(x)\cdot g(x). \end{aligned}$$

Analogously follows from (7)

$$
\begin{aligned}
& f(x+\Delta x)\cdot g(x+\Delta x)-f(x)\cdot g(x)\le \\
& \quad \le f(x+\Delta x)\cdot(g(x+\Delta x)-g(x))+g(x+\Delta x)\cdot(f(x+\Delta x)-f(x))\le \\
& \quad \le f(x+\Delta x)\cdot D_g(x+\Delta x)\cdot\Delta x+g(x+\Delta x)\cdot D_f(x+\Delta x)\cdot\Delta x .
\end{aligned}
$$

Altogether we get

$$
\begin{aligned}
& (f(x)\cdot D_g(x)+g(x)\cdot D_f(x))\cdot\Delta x\le \\
& \quad \le f(x+\Delta x)\cdot g(x+\Delta x)-f(x)\cdot g(x)\le \\
& \quad \le (f(x+\Delta x)\cdot D_g(x+\Delta x)+g(x+\Delta x)\cdot D_f(x+\Delta x))\cdot\Delta x .
\end{aligned}
$$

Thus follows the *product rule*.

$$D_{f\cdot g}(x)=f(x)\cdot D_g(x)+D_f(x)\cdot g(x). \quad \textit{product rule} \tag{9}$$

For example the derivative of becomes sin·cos

$$
\begin{aligned}
D_{\sin\cdot\cos}(x) &= \sin x\cdot D_{\cos}(x)+D_{\sin}(x)\cdot\cos x=\sin x\cdot -\sin x+\cos x\cdot\cos x = cos^2 x - sin^2 x = \\
&= 1-2\cdot\sin^2 x.
\end{aligned}
$$

5.1.2 Partial Integration

This rule can be used to integrate by reordering

$$f(x)\cdot D_g(x)=D_{f\cdot g}(x)-g(x)\cdot D_f(x).$$

If the integral of $g(x)\cdot D_f(x)$ is known, then we can integrate $f(x)\cdot D_g(x)$.

$$I_{f(x)\cdot D_g(x)}=f(x)\cdot g(x)-I_{g(x)\cdot D_f(x)}. \quad \textit{Partial Integration} \tag{10}$$

E.g. $f(x) = x$, $g(x) = -\cos x$, then $D_f(x) = 1$, $D_g(x) = \sin x$, and $I_{\cos}(x) = \sin x$, thus follows

$$I_{x\cdot\sin x}(x)=-x\cdot\cos x-I_{-\cos}(x)=-x\cdot\cos x+\sin x .$$

5.2 Reciprocal rule

The reciprocal rule determines the derivative of the reciprocal $1/f$ of a function f. Let f be positive and never zero: $f(x) > 0$. Let $1/f$ be monotonously ascending. We have the relationship

$$
\begin{aligned}
& (f(x+\Delta x)-f(x))^2=f(x+\Delta x)\cdot f(x+\Delta x)-2\cdot f(x+\Delta x)\cdot f(x)+f(x)\cdot f(x)\ge 0, \\
& f(x+\Delta x)\cdot(f(x+\Delta x)-f(x))\ge f(x+\Delta x)\cdot f(x)-f(x)\cdot f(x), \\
& -\frac{f(x+\Delta x)-f(x)}{f^2(x)}\le\frac{1}{f(x+\Delta x)}-\frac{1}{f(x)} .
\end{aligned}
$$

Analogous follows

$$
\begin{aligned}
& (f(x+\Delta x)-f(x))^2=f(x+\Delta x)\cdot f(x+\Delta x)-2\cdot f(x+\Delta x)\cdot f(x)+f(x)\cdot f(x)\ge 0, \\
& f(x+\Delta x)\cdot f(x+\Delta x)-f(x+\Delta x)\cdot f(x)\ge f(x+\Delta x)\cdot f(x)-f(x)\cdot f(x), \\
& \frac{1}{f(x+\Delta x)}-\frac{1}{f(x)}\ge-\frac{f(x+\Delta x)-f(x)}{f^2(x+\Delta x)} .
\end{aligned}
$$

Altogether we get

$$-\frac{f(x+\Delta x)-f(x)}{f^2(x)}\leq\frac{1}{f(x+\Delta x)}-\frac{1}{f(x)}\leq-\frac{f(x+\Delta x)-f(x)}{f^2(x+\Delta x)}.$$

The reciprocal rule follows

$$D_{1/f}(x)=\frac{-D_f(x)}{f^2(x)}.\quad \textit{Reciprocal rule} \tag{11}$$

As an example for the reciprocal rule (12) let us compute the derivative with negative exponent. Let be $f(x) = x^k$, where $k > 0$. then

$$D_{x^{-k}}(x)=D_{1/f}(x)=\frac{-D_f(x)}{f^2(x)}=\frac{-D_{x^k}(x)}{x^{2k}}=\frac{-k\cdot x^{k-1}}{x^{2k}}=-k\cdot x^{-k-1}.$$

Die power rule () holds for negative integer exponents as well. (12)

5.3 Quotient rule

The product rule follows from the product of a function f and the reciprocal of a function g.

$$D_{f/g}(x)=D_f(x)/g(x)+f(x)\cdot D_g(x)=\frac{D_f(x)}{g(x)}-\frac{f(x)\cdot D_g(x)}{g^2(x)}=\frac{D_f(x)\cdot g(x)-f(x)\cdot D_g(x)}{g^2(x)}.$$

The quotient rule is therefore

$$D_{f/g}(x)=\frac{D_f(x)\cdot g(x)-f(x)\cdot D_g(x)}{g^2(x)}.\quad \textit{Quotient rule} \tag{13}$$

As an example for the quotient rule (13) we derive the quotient of sine and cosine, that is the tangent.

$$D_{\sin/\cos}(x)=\frac{D_{\sin}(x)\cdot\cos x-\sin x\cdot D_{\cos}(x)}{\cos^2 x}=\frac{\cos x\cdot\cos x-\sin x\cdot -\sin x}{\cos^2 x}=\frac{1}{\cos^2 x}.$$

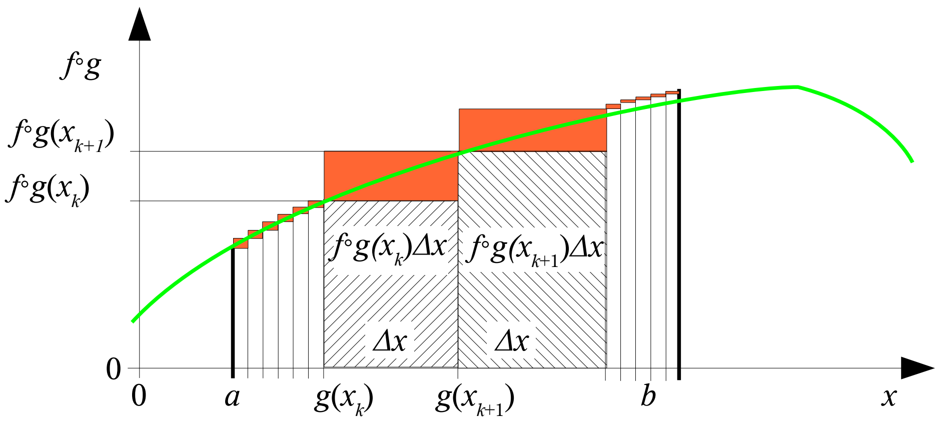
As another example let us compute the derivative of $\sin(x)/x$

$$D_{\sin x/x}(x)=\frac{x\cdot D_{\sin}(x)-D_x(x)\cdot\sin x}{x^2}=\frac{x\cdot\cos x-\sin x}{x^2}=\frac{\cos x}{x}-\frac{\sin x}{x^2}.$$

5.4 Substitution rule

The substitution rule analyses functions like $f(g(x))$; this can be written as $f{\circ}g(\mathrm{x})$, where $f{\circ}g$ can be looked at as a new function h: $f(g(x)) = f{\circ}g(x) = h(x)$. For functions the presentations $f(g(x))$ and $f{\circ}g(x)$ are equivalent. However, if $f(g(x))$ will be integrated, intregration is be done for the variable x, so that the formula for I_f cannot be used as a primitive for $f(g(x))$. The formula $I_f(g(x))$ means the function I_f with parameter $g(x)$, while $I_{f\circ g}(x)$ means the primitive of $f{\circ}g$, what can be completely different from $I_f(g(x))$.

From the diagram



follows

$$f(g(x))\cdot\big(g(x+\Delta x)-g(x)\big)\leq I_f(g(x+\Delta x))-I_f(g(x))\leq \\ \leq f(g(x+\Delta x))\cdot\big(g(x+\Delta x)-g(x)\big). \tag{14}$$

Using the definition of a derivative (8) yields

$$f(g(x))\cdot D_g(x)\cdot\Delta x\leq f(g(x))\cdot\big(g(x+\Delta x)-g(x)\big)\cdot\Delta x\leq \\ \leq I_f(g(x+\Delta x))-I_f(g(x))\leq \\ \leq f(g(x+\Delta x))\cdot\big(g(x+\Delta x)-g(x)\big)\cdot\Delta x\leq \\ \leq f(g(x+\Delta x))\cdot D_g(x+\Delta x)\cdot\Delta x. \tag{15}$$

This rule states that the primitive of the function $f{\circ}g{\cdot}D_g$ is the primitive of f with parameter $g(x)$. Thus from (15) follows the substitution rule for integration

$$I_{f\circ g\cdot D_g}(x)=I_f(g(x)).\quad \textit{Substitution rule for Integration} \tag{16}$$

For example let be $g(x) = a{\cdot}x{+}b$, then $D_g(x) = a$, thus $I_f(a{\cdot}x{+}b)$ is the integral of $f{\circ}g(x){\cdot}a$, or because of linearity of integration follows by reducing

$$I_{f(a\cdot x+b)}(x)=I_{f\circ g}(x)=\frac{1}{a}\cdot I_f(a\cdot x+b).$$

Substitution rule for integration is possible only for very special functions. More general rules follow for differentiation.

5.4.1 Substitution rule for differentiation

For the derivative we assume that the functions D_f and D_g are given, where

$$D_f(x)\cdot\Delta x\leq f(x+\Delta x)-f(x)\leq D_f(x+\Delta x)\cdot\Delta x, \\ D_g(x)\cdot\Delta x\leq g(x+\Delta x)-g(x)\leq D_g(x+\Delta x)\cdot\Delta x.$$

From (14) follows after renaming(f becomes D_f, I_f becomes f)

$$D_f(g(x))\cdot D_g(x)\cdot\Delta x\leq D_f(g(x))\cdot\big(g(x+\Delta x)-g(x)\big)\leq \\ \leq f(g(x+\Delta x))-f(g(x))\leq \\ \leq D_f(g(x+\Delta x))\cdot\big(g(x+\Delta x)-g(x)\big)\leq D_f(g(x+\Delta x))\cdot D_g(x+\Delta x)\cdot\Delta x.$$

Thus follows the substitution rule differentiation

$$D_{f\circ g}(x)=D_f(g(x))\cdot D_g(x).\quad \textit{Substitution rule for differentiation} \tag{17}$$

Technically, $g(x)$ is replaced by a variable $z = g(x)$ and then the derivative of f to z is computed. This is to be multiplied by the derivative of z to x to get the derivative of $f(g(x))$. Pseudoformally the is justified by the differential quotient

$$D_{f\circ g}(x)=D_f(z)\cdot D_z(x)=\frac{d\,f}{d\,z}\cdot\frac{d\,z}{d\,x}=\frac{d\,f}{d\,x}.$$

In this formula the differential dz reduce. However, this is no consistent definition.

For example for any function f and $a(\mathrm{x}) = a\cdot x+b$ follows as derivative of $f(a(x)) = f(a\cdot x+b)$

$$D_{f\circ a}(x)=D_f(a(x))\cdot D_a(x)=D_f(a\cdot x+b)\cdot a=a\cdot D_f(a\cdot x+b).$$

5.5 Inversion rule

Let be g the inverse function of f, $f(g(x)) = x$ for each x; sometimes the notation f^{1} is used for the inverse function f. The inverse function is defined uniquely only, when f does never attain the same value twice in the interval considered, thus $f(x) \neq f(y)$ if $x \neq y$. Thus only strictly monotonous functions are considered here.

5.5.1 Inversion rule for differentiation

The inversion rule for derivatives follows simply from the substitution rule. From substitution rule (17) follows with $D_{f\circ g}(x) = D_x(\mathrm{x}) = 1$, that $D_{f\circ g}(x) = D_f(g(x))\cdot D_g(\mathrm{x}) = 1$, so that we have

$$D_g(x)=\frac{1}{D_f(g(x))}. \quad \text{Inversion ruel} \tag{18}$$

For example let be $f(x) = e^x$, then $f^{1}(x) = \ln x$, thus

$$D_{\ln}(x)=\frac{1}{D_f(\ln x)}=\frac{1}{e^{\ln x}}=\frac{1}{x}.$$

Another application of inverse functions is that for $f(x) = x^k$, which is obviously $g(x) = x^{1/k}$.

$f(g(x)) = (g(x))^k = (x^{1/k})^k = x^{k/k} = x^1 = x.$

Then we get for the derivative

$$D_{x^{1/k}}(x)=\frac{1}{D_{x^k}(x^{1/k})}=\frac{1}{k\cdot(x^{1/k})^{k-1}}=\frac{1}{k}\cdot x^{(1-k)/k}=\frac{1}{k}\cdot x^{1/k-1}.$$

Thus the power rule () follows also for rational exponents. As can be shown by substitution rule this holds for all rational exponents. For arbitrary integer numbers m and k ($k\neq 0$) follows

$$D_{x^{m/k}}(x)=D_{(x^{1/k})^m}(x)=D_{(x)^m}(x^{1/k})\cdot D_{x^{1/k}}(x)=m\cdot(x^{1/k})^{m-1}\cdot\frac{1}{k}\cdot x^{1/k-1}=\frac{m}{k}\cdot x^{(m-1)/k+1/k-1}=\frac{m}{k}\cdot x^{m/k-1}.$$

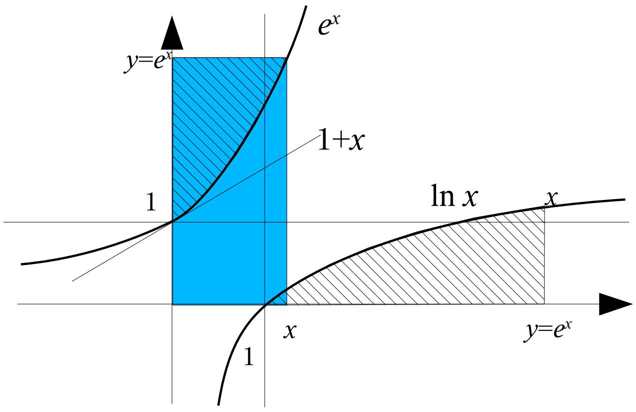
Thus the power rule () is valid for all (rational) exponents.

$$\begin{aligned} &D_{x^q}(x)=q\cdot x^{q-1},\\ &I_{x^q}(x)=\frac{1}{q+1}\cdot x^{q+1}, \end{aligned} \quad \textit{Power rule} \tag{19}$$

for each rationale q, with exception $q = -1$, the primitive of which is $\ln x$.

5.5.2 Inversion rule for integration

Another rule follows from simple geometric considerations.



The diagram shows the principal plot of the e-function and the natural logarithm, where the striped areas have the same size. In general follows for an arbitrary monotonously ascending function f, with value $b = f(0)$ at origin, that the area in the interval $[b,y]$ is, where $y = f(x)$, $g(y) = x$,

$$I_g(y)-I_g(b)=x\cdot f(x)-(I_f(x)-I_f(0))=g(y)\cdot y-(I_f(g(y))-I_f(0)),$$

or after renaming of y to x and dropping constant terms we get as primitive function

$$I_g(x)=g(x)\cdot x-I_f(g(x)). \quad \text{Inversion rule for primitive} \tag{20}$$

To show this algebraically, we use rules for addition and substitution (16)

$$I_g(x)+I_f(g(x))=I\left[g(x)+f(g(x))\cdot D_g(x)\right]=I\left[g(x)+x\cdot D_g(x)\right].$$

Here $I[f](x)$ stands for $I_f(x)$, to keep the formulae clear. Since we assume $f(g(x)) = x$ follows from product rule $D_{g\cdot h}(x) = D_g(x)\cdot h(x)+D_h(x)\cdot g(x)$, with $h(x) = x$ and $D_x(x) = 1$

$$g(x)\cdot x=I[D_{g\cdot x}](x)=I\left[g(x)+x\cdot D_g(x)\right](x)=I_g(x)+I_f(g(x)).$$

This proves the inversion rule for primitives (20) algebraically. As proved already, follows for the natural logarithm

$$I_{\ln}(x)=\ln(x)\cdot x-I_{e^x}(\ln(x))=\ln(x)\cdot x-e^{\ln(x)}=\ln(x)\cdot x-x.$$

Other applications will be given below, for the inverse functions of trigonometric function.

5.5.3 Quotient rule for primitives

In very special cases also integration can be done by a quotient rule. The primitive of a function that ca be displayed as $D_g(x)/g(x)$ is ln(g(x)). We use substitution rule (16)

$$I_{f\circ g\cdot D_g}(x)=I_f(g(x)).$$

If $f(x) = 1/x$, with primitive $I_f = \ln x$, then the primitive of

$$\frac{D_g(x)}{g(x)}=D_g(x)\cdot\frac{1}{g(x)}=D_g(x)\cdot f\circ g(x)$$

is the function

$$I_{D_g/g}(x)=I_{D_g\cdot f\circ g}(x)=I_f(g(x))=I_{1/x}(g(x))=-\ln(g(x)). \tag{21}$$

5.6 Tangent and Cotangent

Tangent and cotangent are defined as

$\tan x = \sin x / \cos x$,

$\cot x = \cos x / \sin x$,

and have a close relationship

$\tan x = 1 / \cot x$.

5.6.1 Integral

The primitive of $\tan x$ is $-\ln(\cos x)$. We consider only the intervall $[0,\pi/2)$, since tangent has a pole for $x = \pi/2$. Because of $\tan -x = -\tan x$ the results hold in the interval $(-\pi/2, \pi/2)$ as well. We use substitution rule (16)

$$I_{f \circ g \cdot D_g}(x) = I_f(g(x)).$$

Is $f(x) = -1/x$, with the integral $I_f = -\ln x$, and $g(x) = \cos x$ with $D_{\cos} = -\sin$, then the primitive of

$$\tan x = \frac{\sin x}{\cos x} = -D_{\cos x} \cdot \frac{1}{\cos x} = D_g(x) \cdot f \circ g(x)$$

is the function

$$I_{\tan}(x) = I_{D_g \cdot f \circ g}(x) = I_f(g(x)) = I_{-1/x}(\cos x) = -\ln(\cos x).$$

Because of $\cot x = \cos x / \sin x$ and $D_{\sin} = \cos$ follows in the same way

$$I_{\cot}(x) = \ln(\sin x).$$

5.6.2 Derivative

The derivative of tangent follows from quotient rule for differentiation.

$$D_{\tan}(x) = D_{\sin/\cos}(x) = \frac{\cos x \cdot \cos x - \sin x \cdot - \sin x}{\cos^2 x} = \frac{1}{\cos^2 x}.$$

Analogous follows for the derivative of the cotangent from the quotient rule for differentiation.

$$D_{\cot}(x) = D_{\cos/\sin}(x) = \frac{-\sin x \cdot \sin x - \cos x \cdot \cos x}{\sin^2 x} = \frac{-1}{\sin^2 x}.$$

This can also be shown by reciprocal rule (11).

$$D_{\cot}(x) = D_{1/\tan}(x) = \frac{-D_{\tan}(x)}{\tan^2 x} = \frac{-1/\cos^2 x}{\sin^2 x / \cos^2 x} = \frac{-1}{\sin^2 x}.$$

5.6.3 Direct Proves

Although many formulae can be proved by rules derived in this section, they can usually proved directly. For example for tangent we have to prove because of our standard definition (,)

$$\frac{1}{\cos^2 x} \cdot \Delta x \le \tan(x + \Delta x) - \tan x \le \frac{1}{\cos^2(x + \Delta x)} \cdot \Delta x.$$

Multiplying the numerators yields

$$\cos^2(x+\Delta x)\cdot\Delta x \le \sin(x+\Delta x)\cdot\cos(x+\Delta x)\cdot\cos^2 x - \sin x\cdot\cos x\cdot\cos^2(x+\Delta x)=$$
$$=\cos x\cdot\cos(x+\Delta x)\cdot(\sin(x+\Delta x)\cdot\cos x - \sin x\cdot\cos(x+\Delta x))=$$
$$=\cos x\cdot\cos(x+\Delta x)\cdot\sin(\Delta x)\le\cos^2 x\cdot\Delta x\,.$$

We used the trigonometric theorem for addition sin ((x+Δx)–x). Dividing the right inequality by the positive term cos x yields

$$\sin\Delta x\cdot\cos(x+\Delta x)\le\cos x\cdot\Delta x\,;$$

this is true since cos(x+Δx) < cos x, and sin Δx < Δx. For the left hand side inequality we have

$$\cos(x+\Delta x)\cdot\Delta x\le\sin\Delta x\cdot\cos x\,.$$

To prove this we use the inequalities for cosine ()

$$\cos(x+\Delta x)\cdot\Delta x\le\sin(x+\Delta x)-\sin x=\sin x\cdot(\cos\Delta x-1)+\cos x\cdot\sin\Delta x\le\cos x\cdot\sin\Delta x\,,$$

since for positive Δx always holds cos Δx < 1.

This shows that any direct prove can become quite more complicated, however, should always be possible, i.e. any formula should be provable by our standard definition (,), as well.

5.6.4 Antitrigonometric functions

We call inverse function of a trigonometric function like sine or cosine the arcus functions: arcsin, arccos, arctan arccot. From formula (18)

$$D_g(x)=\frac{1}{D_f(g(x))}$$

follows for arcsin

$$D_{\arcsin}(x)=\frac{1}{D_{\sin}(\arcsin(x))}=\frac{1}{\cos(\arcsin(x))}=\frac{1}{\sqrt{1-\sin^2(\arcsin(x))}}=\frac{1}{\sqrt{1-x^2}}\,,$$

and for arctan

$$D_{\arctan}(x)=\frac{1}{D_{\tan}(\arctan(x))}=\cos^2(\arctan(x))=\frac{1}{\sqrt{1+\tan^2(\arctan(x))}}=\frac{1}{\sqrt{1+x^2}}\,.$$

Also primitives follow from inversion rule (20), that read

$$I_g(x)=g(x)\cdot x-I_f(g(x)).$$

For the primitive of arcsin we get

$$I_{\arcsin}(x)=\arcsin(x)\cdot x-I_{\sin}(\arcsin(x))=\arcsin(x)\cdot x+\cos(\arcsin(x))=\arcsin(x)\cdot x+\sqrt{1-x^2}\,.$$

For the primitive of arccos we get

$$I_{\arccos}(x)=\arccos(x)\cdot x-I_{\cos}(\arcsin(x))=\arccos(x)\cdot x-\sin(\arccos(x))=\arccos(x)\cdot x-\sqrt{1-x^2}.$$

For primitive of arctan we get

$$I_{\arctan}(x)=\arctan(x)\cdot x-I_{\tan}(\arctan(x))=\arctan(x)\cdot x+\ln(\cos(\arctan(x))).$$

The last formula can be simplified by some trigonometric relationships. We have

$$\tan x = \frac{\sin x}{\cos x} = \frac{\sqrt{1-\cos^2 x}}{\cos x},$$
$$\tan^2 x \cdot \cos^2 x = 1 - \cos^2 x,$$
$$\cos x = \frac{1}{\sqrt{1+\tan^2 x}}.$$

and then follows

$$\cos \arctan x = \frac{1}{\sqrt{1+x^2}},$$

if arctan is used as parameter. Since logarithm of a root is the logarithm of the half value, and reciprocal value is the negative logarithm we get as primitive of acrtan

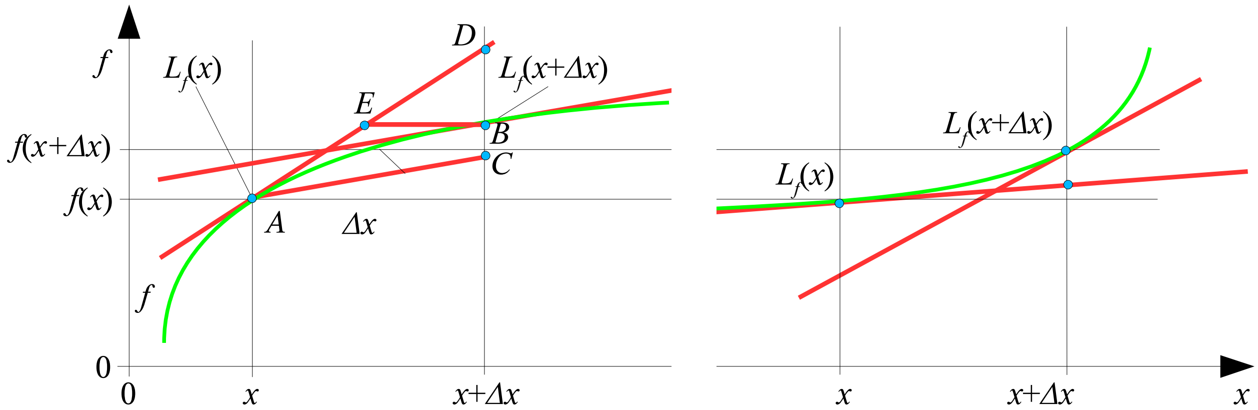
$$I_{\text{arctan}}(x) = \arctan(x) \cdot x - \frac{1}{2} \cdot \ln(1+x^2).$$

6 Length of a curve

To find the length of a curve the standard mathematics derives the formala

$$L_f(z)=\int_0^z \sqrt{1+D_f^2(x)}\ dx$$

We will derive this formala by the inequalities (,). The following diagram for a monotonously ascending curve that is convex from top shows that the length *AC* is shorter then the (green) curve of the function *f* from *A* to *B*, since the length *AC* is shorter than the secant from *A* to *B*. The length AC can be computed from *Δx*/cos α, where α is the angle of the tangent in *B* at the curve of the function *f* in point *x*+*Δx*. Analogous the length AD is longer than the curve of the function *f* from *A* to *B*, since the length *AE*+*EB* is always longer than then this curve and itself is longer than *AD*. The length AD can be computed from *Δx*/cos β, where β is the angle of the tangent in *A* at the curve of the function *f* in point *x*.



If the curve is convex from the bottom follows analogously that *Δx*/cos β is shorter than the curve between two points, when β is the angle of the tangent to the curve of the function *f* in point *x*, or that *Δx*/cos α is longer than the curve between two points if α is the angle of the tangent at the curve of the function *f* in point *x*+*Δx*. Again the length of the curve can be limited between two straight lines, und again the total length can be found by the sum of the partial lengths. To determine the deviation, the differences of the straight lines are to be summed, thus

$$\sum_{i=0}^{N-1} \Delta x/\cos\alpha_{i+1}-\Delta x/\cos\alpha_i=\Delta x/\cos\alpha_N-\Delta x/\cos\alpha_0=\Delta x\cdot\left(\frac{1}{\cos\alpha_N}-\frac{1}{\cos\alpha_0}\right).$$

The deviation is proportional to the step width *Δx*, where we only have to assume that the angle of the ascent is less than 90°.

The length *Δx*/cos α can be found from the tangent of the angle α of the tangent to the horizontal line, where we use the identity

$$\cos x=\frac{1}{\sqrt{1+\tan^2 x}}$$

and the length of the tangent in the section $[x, x+\Delta x]$ is

$$L_t = \frac{\Delta x}{\cos x} = \Delta x \cdot \sqrt{1+\tan^2 x} = \Delta x \cdot \sqrt{1+D_f^2(x)}.$$

For a curve that is convex from the bottom follow the inequalities

$$\Delta x \cdot \sqrt{1+D_f^2(x)} \le L_f(x+\Delta x) - L_f(x) \le \Delta x \cdot \sqrt{1+D_f^2(x+\Delta x)}. \tag{22}$$

For a curve that is convex from the top follow the inequalities

$$\Delta x \cdot \sqrt{1+D_f^2(x+\Delta x)} \le L_f(x+\Delta x) - L_f(x) \le \Delta x \cdot \sqrt{1+D_f^2(x)}. \tag{23}$$

We can assume that L_f is the primitive to $\sqrt{1+D_f^2(x)}$ and come to the same result as in standard mathematics. If there is a deviation between the real length of the curve and the function L_f follows from reducing the step with Δx that the function L_f does not obey (22) and (23) from which uniqueness of length function L_f follows.

6.1.1 Example

The function $f(x) = a \cdot x$ has the derivative $D_f(x) = a$. Then its length in interval $[0,x]$ is given by the function

$$L_f(x) = \sqrt{1+a^2} \cdot x.$$

Then follows with the first (and analogous the second) inequalities

$$\begin{aligned}
&\Delta x \cdot \sqrt{1+D_f^2(x)} = \Delta x \cdot \sqrt{1+a^2} \le \\
&\quad \le L_f(x+\Delta x) - L_f(x) = \sqrt{1+a^2} \cdot (x+\Delta x) - \sqrt{1+a^2} \cdot x = \sqrt{1+a^2} \cdot \Delta x \le \\
&\quad \le \Delta x \cdot \sqrt{1+D_f^2(x+\Delta x)} = \Delta x \cdot \sqrt{1+a^2}.
\end{aligned}$$

6.1.2 Example

The function

$$f(x) = \frac{2}{3} \cdot x^{3/2}$$

hat the derivative $D_f(x) = x^{1/2}$. Then its length in the interval $[0,x]$ is given by the function

$$L_f(x) = \frac{2}{3} \cdot (1+x)^{3/2}.$$

Since this function is monotonously ascending and convex from the bottom inequalities (22) must hold.

$$\begin{aligned}
&\Delta x \cdot \sqrt{1+D_f^2(x)} = \Delta x \cdot \sqrt{1+x} \le \\
&\quad \le L_f(x+\Delta x) - L_f(x) = \frac{2}{3} \cdot (1+x+\Delta x)^{3/2} - \frac{2}{3} \cdot (1+x)^{3/2} \le \\
&\quad \le \Delta x \cdot \sqrt{1+D_f^2(x+\Delta x)} = \Delta x \cdot \sqrt{1+x+\Delta x}.
\end{aligned}$$

We substitute $z = 1+x$ and get for the left inequality

$$\Delta x\cdot\sqrt{z}=\Delta x\cdot z^{1/2}\le\frac{2}{3}\cdot(z+\Delta x)^{3/2}-\frac{2}{3}\cdot z^{3/2},$$
$$3\cdot\Delta x\le 2\cdot\sqrt{\frac{(z+\Delta x)^3}{z}}-2\cdot z,$$
$$3\cdot\Delta x+2\cdot z\le 2\cdot\sqrt{\frac{(z+\Delta x)^3}{z}},$$
$$9\cdot\Delta^2 x+4\cdot z^2+12\cdot z\cdot\Delta x\le 4\cdot\frac{(z+\Delta x)^3}{z}=4\cdot(z^2+3\cdot z\cdot\Delta x+3\cdot\Delta^2 x+\frac{\Delta^3 x}{z}),$$
$$0\le 3\cdot\Delta^2 x+4\cdot\frac{\Delta^3 x}{z}=3\cdot\Delta^2 x+4\cdot\frac{\Delta^3 x}{1+x}.$$

for positive x and Δx this inequality holds. For the right inequality we must prove that

$$\frac{2}{3}\cdot(1+x+\Delta x)^{3/2}-\frac{2}{3}\cdot(1+x)^{3/2}=\frac{2}{3}\cdot(z+\Delta x)^{3/2}-\frac{2}{3}\cdot z^{3/2}\le\Delta x\cdot\sqrt{z+\Delta x},$$
$$2\cdot(z+\Delta x)^{3/2}-2\cdot z^{3/2}\le 3\cdot\Delta x\cdot\sqrt{z+\Delta x},$$
$$2\cdot(z+\Delta x)-2\cdot\sqrt{\frac{z^3}{z+\Delta x}}\le 3\cdot\Delta x,$$
$$2\cdot(z+\Delta x)-3\cdot\Delta x=2\cdot z-\Delta x\le 2\cdot\sqrt{\frac{z^3}{z+\Delta x}}.$$

If $\Delta x > 2\cdot z = 2+2\cdot x$, then the left hand side is negative, thus the inequality is correct. Thus if $\Delta x \le 2\cdot z$ then both sides are positive and we can square.

$$4\cdot z^2+\Delta^2 x-4\cdot z\cdot\Delta x\le 4\cdot\frac{z^3}{z+\Delta x},$$
$$4\cdot z^3+z\cdot\Delta^2 x-4\cdot z^2\cdot\Delta x+4\cdot z^2\cdot\Delta x+\Delta^3 x-4\cdot z\cdot\Delta^2 x=4\cdot z^3+\Delta^3 x-3\cdot z\cdot\Delta^2 x\le 4\cdot z^3,$$
$$\Delta x\le 3\cdot z.$$

Since we assumed $\Delta x \le 2\cdot z < 3\cdot z$ follows the inequality. Thus the inequality is valid for all positive z and Δx.

7 Stieltjes-Integral

Eine Erweiterung des Integralbegriffs ist das sogenannte *Stieltjes*-Integral, welches das Integral I_{fg} einer Funktion f über eine andere Funktion g folgendermaßen definiert:

$$\begin{aligned} & f(x) \cdot \big(g(x+\Delta x) - g(x)\big) \leq \\ & \qquad I_{fg}(x+\Delta x) - I_{fg}(x) \leq \\ & \qquad \leq f(x+\Delta x) \cdot \big(g(x+\Delta x) - g(x)\big). \end{aligned} \tag{24}$$

Ist g differenzierbar, so lässt sich mit der Definition der Ableitungen (8) herleiten

$$\begin{aligned} & f(x) \cdot D_g(x) \cdot \Delta x \leq f(g(x)) \cdot \big(g(x+\Delta x) - g(x)\big) \leq \\ & \qquad \leq I_{fg}(x+\Delta x) - I_{fg}(x) \leq \\ & \qquad \leq f(x+\Delta x) \cdot \big(g(x+\Delta x) - g(x)\big) \leq \\ & \qquad \leq f(x+\Delta x) \cdot D_g(x+\Delta x) \cdot \Delta x. \end{aligned} \tag{25}$$

Also gilt für differenzierbares g

$$I_{fg} = I_{f \cdot D_g}.$$

Ist g jedoch nicht differenzierbar, so lässt sich eine solche Gleichung nicht herleiten.

Die Bedeutung dieses Integrals liegt darin, dass es den sogenannten Erwartungswert E[f] einer Funktion einfacher zu definieren gestattet. Wenn $f(x)$ den Wert einer Zufallszahl darstellt und $p(x)$ die entsprechende Wahrscheinlichkeit für das Auftreten des Werts $f(x)$, dann kann man definieren

$$E[f] = I_{fp}\Big]_0^1$$

Existiert die Ableitung von p, so wird D_p auch als *Dichte* von p bezeichnet.

Ist $g(x) = x$, so ist $D_x = 1$, also wie beim normalen Integral $I_{fx} = I_f$.

8 Literature

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