Integral Theory

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1 Preface

This report introduces integration theory in a much simpler way than usually presented, since only pure algebraic methods are used. The goal of integration is to find the area beneath a curve. We use a relatively simple algebraic expression to define the properties of 'primitive', which are sufficient and necessary to specify that primitive uniquely.

Our method does not require theory of limits of sequences or functions, so it may be used to introduce the interesting and important theory of integration much earlier in school or university than this is usually done.

This paper is a direct translation from the original paper, written in German. Although they are almost identical, there can be some differences, since the German paper is more often upgraded.

1.1 Summary

This paper introduces a completely new method for integration, that specifies all integrable functions by an algebraic approach. The set of integrable functions is the set of all functions that are piece-wise monotonous, which means that there are intervals $[u_i,v_i]$, where $u_i < v_i$, in which the function being integrable is either monotonously ascending or monotonously descending.

Specification of integral function I_f of the function f that is monotonously ascending in interval $[u,v]$, is done only by the set of inequalities

$$f(x)\cdot\Delta x \;\le\; I_f(x+\Delta x)-I_f(x) \;\le\; f(x+\Delta x)\cdot\Delta x, \text{ for all } x, \Delta x \text{ where } u\le x<x+\Delta x\le v;$$

and for function f that is monotonously descending in interval $[u,v]$ by the set of inequalities

$$f(x)\cdot\Delta x \;\ge\; I_f(x+\Delta x)-I_f(x) \;\ge\; f(x+\Delta x)\cdot\Delta x, \text{ for all } x, \Delta x \text{ where } u\le x<x+\Delta x\le v.$$

The paper proves that these are necessary and sufficient conditions for I_f to be integral function of f; this means that all integral functions I_f of f must fulfil those conditions, while all functions I_f fulfilling those conditions are integral functions of f. Thus integral functions can be specified and verified by those inequalities, what is much simpler than computing lower and upper sums and their limits.

The main goal of this report is to demonstrate that this technique can introduce integration as is well known from standard mathematics, without use of analytical techniques like infinite sums, convergence, limits etc. The class of integrable functions encloses also some that are for example not Cauchy integrable, like the pulse function.

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2 Introduction

Integration is used to determine the area beneath a function. This difficult method can be solved easily with the method introduced in this paper. To achieve this, for a function f another function I_f is found, that determines the requested area. We analyse the properties of the function I_f and show how these properties can be stated mathematically. Although our method is much simpler, it is as powerful as e.g. Cauchy- or Riemann-Integration. We do neither need limits or general summation formulae, the latter of which may be proved by our approach.

Determination of areas has been an incessant question in physics and mathematics, solved partially by antique mathematicians like Archimedes [Luca], however answered more generally by Newton and Leibniz only centuries ago. There are several difficulties with these approaches, which seem to be solved only some centuries after the inventors. The mathematical foundation are very difficult to understand and to prove.

This report gives a much simpler introduction in this method. We start with some basic mathematical requirements, numbers, algebra, computation with inequalities and functional notions.

Then we state the very simple conditions, from which follows the integral (or areal) function and its main properties. The most important property is that this areal function is unique, so that these conditions are sufficient to develop a complete theory of integration. Differentiation is introduced as inverse of integration, since it is required for some rules to be developed later.

Next primitives of some general functions are proved, besides polynomials, trigonometric functions as well as exponentiation and natural logarithm. We can prove these results directly, not via sequence development or general summation. This makes our method much simpler than in standard mathematics.

Then some rules and formulae are introduced, to derive more primitives in a simpler way, where differentiation and some rules are required. Finally we give a hint how to apply this method to other questions, like the computation of the length of a curve.

This report shall show a canonical way to calculus. This should help to present this interesting field of mathematics in an earlier stage in education, where the usual preparations like limits, general sums or special properties of real numbers are not required. Also 'infinitesimal' elements are not used. We start with integration instead of differentiation, since calculation of areas seems to be much more interesting than ascents of functions.

We use a slightly different notation than standard mathematics use, since this is easier to understand; this has only didactical reasons and may be replaced by standard notation if this seems to become necessary. A primitive of f is always called I_f (instead of F, as is used in many references), derivation is written as D_f, instead of f' oder df/dx (the notation $D(f(x))$ for derivation is used by Dieudonné, for example).

As sources we use besides some standard references, also internet links, which are easier availabe.

3 Fundamentals

3.1 Numbers

We use numbers, like rational numbers or real numbers with the usual properties of an ordered commutative field. Operations are add and minus, as well as multiplication and division.

3.2 Inequalities

When two expressions yield the same result we express this with equality sign '='. If one expression a can be less than another b, then we use the symbols '<', '≤', '>', '≥'. The expression

$a \leq b$

means that a *is* less or equal to b, e.g. $3 \leq 4$, or $4 \leq 4$, but not $5 \leq 4$.

$a < b$

means that a is less than b, $3 < 4$, but not $4 < 4$,. Analogous statements can be applied to the other symbols.

3.2.1 Algebra of inequalities

For any constant term c we have

when $a < b$, then $a+c < b+c$.

If q is positive, then

if $a < b$, then $a \cdot q < b \cdot q$;

if q is negative, then

if $a < b$, then $a \cdot q > b \cdot q$.

For example, if $2 < 3$, then (with $a = -1$) $-2 > -3$. Analogous holds for reciprocal values:

if $a < b$, then $1/a > 1/b$.

For example, if $2 < 4$, then $½ = 0{,}5 > ¼ = 0{,}25$.

Similar statements holds for other function, e.g. exponential function p^x, so that

if $a < b$, then $p^a < p^b$.

You can add inequalities:

if $a < b$, $c < d$ then $a+c < b+d$.

If a and c are positive, then

if $0 < a < b$, $0 < c < d$ then $a \cdot c < b \cdot c$.

You cannot subtract inequalities

if $1 < 3$, $4 < 9$, then not $1-4 = -3 < 3-9 = -6$.

You cannot divide inequalities

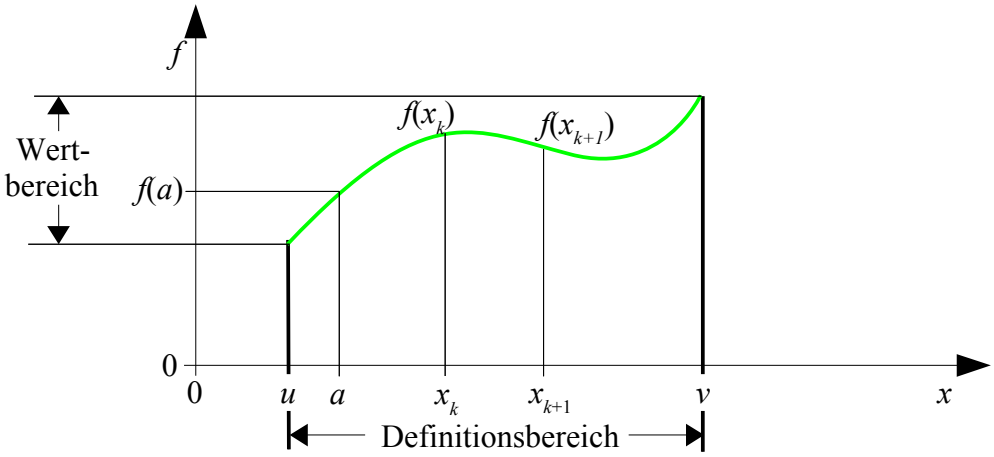
if $1 < 3$, $4 < 9$, then not $4/1 = 4 < 9/3 = 3$.

3.3 Functions

A function f maps a number a to another number, which is written as $f(a)$; f means the function at all, while $f(a)$ is a number from the range of values of the function. The set of numbers a function is

defined for is called range of definition. In $f(a)$ the expression a is called the *argument* or sometimes (*independent*) *variable*, while $f(a)$ is called functional value. The expression $f(x) = x^2$ defines a function with name f, here as example a parabola.

The range of a definition of a function is often restricted for pragmatic reasons. Functions can be displayed in diagrams, where a part of the range of definition is plotted horizontally, while the functional values are plotted vertically. The points $(a,f(a))$ can be connected to a curve. This diagram is called Cartesian system of coordinates. It defines an area beneath that curve, the size of which we will compute.



The horizontal axis is usually called abscissa (lat.: *abscissa* = for 'cut', since only a part of the range of definition is used), while the vertical axis is called *ordinate*. Functional arguments are usually written x_k, x_{k+1} ..., that's why the abscissa is called x-axis, as well. If the values in the ordinate are written as y_k, y_{k+1} ..., this is called y-axis.

A function is called *monotonously ascending*, if $f(x+c) \geq f(x)$ holds for each positive c; it is called *monotonously descending*, if $f(x+c) \leq f(x)$ holds for each positive c. In the first case the function is never descending, in the other case never ascending. Monotonously does mean the values can be constant, i.e. $f(x+c) = f(x)$ holds for some x and positive c. If the function is always ascending or descending, then the function is called *strictly monotonously* ascending or descending.

If a function is in some ranges $[u,v]$ monotonously ascending, in the range $[v,w]$ monotonously descending, then the range of definition is usually restricted to those intervals and the function analysed in the corresponding sections, since monotony is required often.

Another property of function is continuity. Let f be a monotonously ascending function in a range and $f(x+\Delta x) - f(x) < k\cdot\Delta x$ for any positive $\Delta x > 0$ and an adequate number k. Then f is called *continuos* in that range. The same holds for monotonously descending function in a range and $f(x)-f(x+\Delta x) < k\cdot\Delta x$ for any positive $\Delta x > 0$ and an adequate number k. Usually this is written as

$|f(x+\Delta x) - f(x)| < k\cdot\Delta x$ for any positive $\Delta x > 0$ and an adequate number k.

This means that the function does not make any 'jumps'. We can formulate it equivalently in another way:

$-k\cdot\Delta x < f(x+\Delta x) - f(x) < k\cdot\Delta x$.

We use this later to prove that '*primitives*' are continuos.

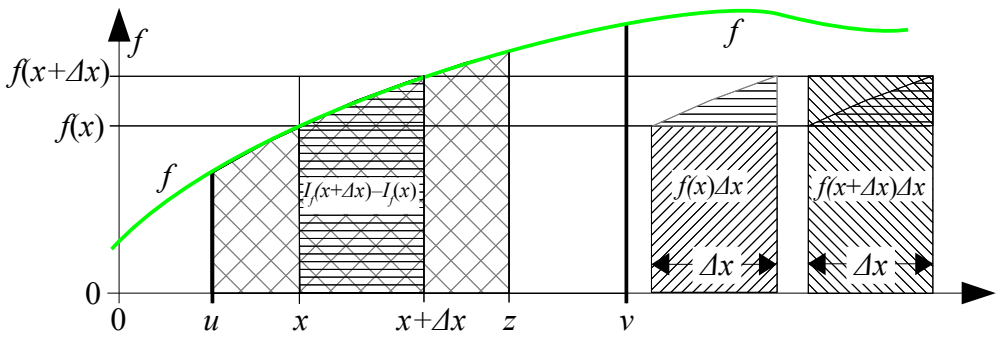
4 The area beneath a function

Our goal is to compute the area beneath a curve. As area we take the geometric area, which can be defined naturally by the sum of the areas of very small rectangles, that are packed densely and covering completely the corresponding area. Since areas beneath general curves cannot be covered by rectangles exactly, this can be an approximation only. However, we will show that the difference must be zero, so that the areal function describes exactly the area beneath a curve.

4.1 Condition for an areal function

In the next diagram the area to be computed is sketched by the squared area. In the interval $[u,v]$ f is monotonously ascending and positive, i.e. $0 \le f(x) \le f(x+\Delta x)$ for all adequate positive Δx, or $\Delta x > 0$. We are looking for the area beneath that function, that is the area restricted by

- the vertical axis u and v,
- the abscissa from $x = u$ to $x = v$,
- and the function f (green drawn in the diagram).



Let us assume there is an *areal function* I_f, determining the area beneath the curve f. Concretely, $I_f(z)$ can be considered to be the area beneath the curve f in the interval $x = 0$ to $x = z$ (the function I_f is not drawn in the diagram); the values are assumed to be from the range of the definition of f and I_f.

We are looking for a condition, that describes the function I_f by use of the function f, where I_f has the following properties: for any x from the range $[u,v]$ and any positive Δx, or $\Delta x > 0$, where $x+\Delta x \le v$ we require

$$f(x)\cdot\Delta x \le I_f(x+\Delta x) - I_f(x) \le f(x+\Delta x)\cdot\Delta x\,. \tag{1}$$

Comparing this condition with the diagram above, the difference

$$I_f(x+\Delta x) - I_f(x)$$

determines the area beneath the curve between the two points x and $x+\Delta x$ on the abscissa. Because of the monotonous property of f a lower bound of this area is the rectangle with height $f(x)$ and size $f(x)\cdot\Delta x$; an upper bound of the area beneath the curve is the rectangle with height $f(x+\Delta x)$ and size $f(x+\Delta x)\cdot\Delta x$. Thus on the one side we have

$$f(x)\cdot\Delta x \le I_f(x+\Delta x) - I_f(x)$$

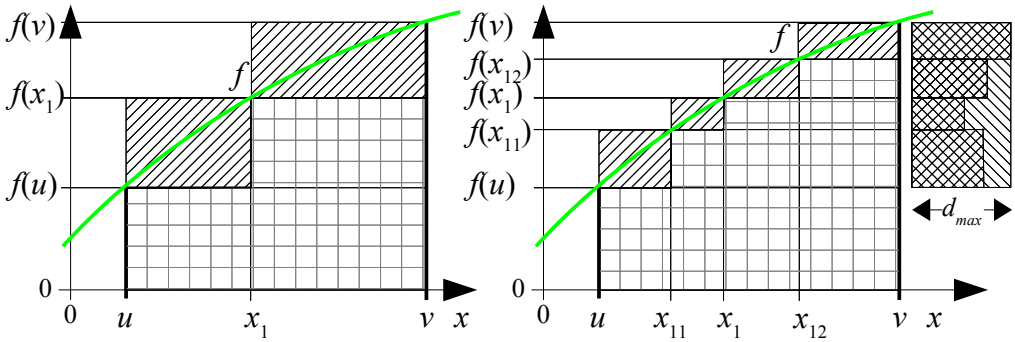
and on the other side holds

$$I_f(x+\Delta x) - I_f(x) \le f(x+\Delta x)\cdot\Delta x\,.$$

This is exactly the meaning of condition (1), expressed with two inequalities.

Next we show that inequalities (1) are sufficient for I_f to be unique. In the next diagram the squared area under the curve f is approximately the area beneath the curve, besides some gappy

holes between the upper borders of the rectangles and the curve.



For the rectangles in the left diagram follow from inequalities (1)

$$f(u)\cdot(x-u)\le I_f(u+(x-u))-I_f(u)=I_f(x)-I_f(u)\le f(u+(x-u))\cdot(x-u),$$
$$f(x)\cdot(v-x)\le I_f(x+(v-x))-I_f(x)=I_f(v)-I_f(x)\le f(x+(v-x))\cdot(v-x).$$

Summing these inequalities yields

$$f(x)\cdot(x-u)+f(x)\cdot(v-x)\le I_f(v)-I_f(u)\le f(x)\cdot(x-u)+f(v)\cdot(v-x).$$

The left hand side is the area of all rectangles beneath the curve, the right hand side is the area of the rectangles 'above' the curve, enclosing the squared areas and the striped areas. Consider that the difference $I_f(v)–I_f(u)$ is independent of the widths of the rectangles; in this example $I_f(v)–I_f(u)$ does not depend on the concrete value of x_1.

Now we decrease the widths of the rectangles by increasing the number of rectangles. In the right diagram the difference between the areas of the rectangles and the areas beneath the curve is much smaller, but from (1) follow the same inequalities as above

$$\begin{array}{lcccl}
f(u)\cdot(x-u) & \le & I_f(x)-I_f(u) & \le & f(x)\cdot(x-u) \\
f(x)\cdot(x-x) & \le & I_f(x)-I_f(x) & \le & f(x)\cdot(x-x) \\
f(x)\cdot(x-x) & \le & I_f(x)-I_f(x) & \le & f(x)\cdot(x-x) \\
f(x)\cdot(v-x) & \le & I_f(v)-I_f(x) & \le & f(v)\cdot(v-x)
\end{array}.$$

Summation of these inequalities yields as well as above

$$\text{area of rectangle beneath curve}\le I_f(v)-I_f(u)\le\text{area of rectangle above curve}.$$

Inequalities (1) mean that these properties hold for <u>each</u> partitioning of the intervals between the borders u and v, since they hold for <u>each</u> positive Δx. In the right diagram the difference between those two areas of rectangles are sketched, which is less or equal to the maximum width of any interval times the difference of the function values $f(v)-f(u)$:

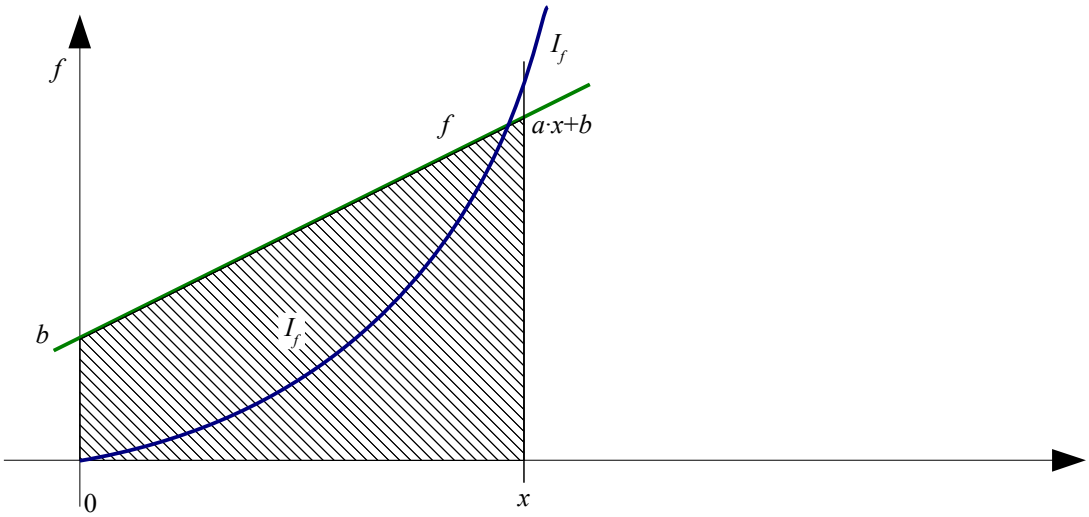
$$\text{maximum difference of upper and lower rectangles}\le (f(v)-f(u))\cdot d_{max}.$$

Since the widths of those rectangles can be made arbitrarily small, the difference must be zero (as will be proved mathematically exact in chapter 5.2 on page 18). Thus function I_f must exactly give the area beneath the curve f.

Thus condition (1) is all we need to prove that I_f is the areal function of f for monotonously ascending functions. We will later show that the same can be proved for all monotonous functions, as well as for all functions the range of definition of which can be partitioned in a finite number of intervals with only ascending or descending function f.

4.1.1 A simple example

As a first very simple example we compute the area beneath a line, that we can compute directly by elementarily geometric considerations. Let be $f(x) = a\cdot x+b$, where $x\geq 0$, $a\geq 0$, $b\geq 0$.



From geometric considerations follows that the area of the trapezoid beneath the line is $I_f(x) = I_{a\cdot x+b}(x) = a/2\cdot x^2+b\cdot x$. To verify this with the inequality (1) we have to show that for this f and I_f the following inequalities hold

$$(a\cdot x+b)\cdot\Delta x\leq a/\ \cdot(x+\Delta x)\ +b\cdot(x+\Delta x)-a/\ \cdot x\ -b\cdot x\leq(a\cdot(x+\Delta x)+b)\cdot\Delta x. \tag{2}$$

We prove this by transforming the inequalities, so that their correctness can be seen easily. At first estimate the squares and cancel common terms.

$$\begin{aligned}(a\cdot x+b)\cdot\Delta x\leq a/\ \cdot(x+\Delta x)\ +b\cdot(x+\Delta x)-a/\ \cdot x\ -b\cdot x=\\ =a/\ \cdot(x\ +\ \cdot x\cdot\Delta x+\Delta\ x)+b\cdot(x+\Delta x)-a/\ \cdot x\ -b\cdot x=\\ =a\cdot x\cdot\Delta x+a/\ \cdot\Delta x\ +b\cdot\Delta x\\ \leq(a\cdot(x+\Delta x)+b)\cdot\Delta x=(a\cdot x+a\cdot\Delta x+b)\cdot\Delta x.\end{aligned}$$

The common positive factor Δx can be cancelled.

$$a\cdot x+b\leq a\cdot x+a/\ \cdot\Delta x+b\leq a\cdot x+a\cdot\Delta x+b.$$

We can subtract the common terms $a\cdot x+b$ and get finally

$$\leq a/\ \cdot\Delta x\leq a\cdot\Delta x.$$

Since the product $a\cdot\Delta x$ is positive, we can cancel.

$$\leq\ /\ \leq\ . \tag{3}$$

Obviously this is correct.

The areal function I_f is called *primitive*.

4.1.2 Exercises

1. Show: if $f(x) = a$, $a > 0$, then $I_f(x) = a\cdot x$. The function is a horizontal line.
2. Show: if $f(x) = a\cdot x + b$, $a < 0$, then $I_f(x) = a/2\cdot x^2+b\cdot x$. Since the function is descending, the inequalities signs must be exchanged.
   Which value has $I_f(x)$, if $b = 0$? Which meaning has I_f in this case?

3. Do you see a general law?
4. Let function f be:

$$f(x)=\begin{cases} x & x\leq \\ -x & x> \end{cases}$$

The function ascends until $x = 1$ and descends then. How can I_f be defined?

4.1.3 A classical example

As a second example we compute a parabolic function. Let be $f(x) = x^2$ and x positive; then we show that $I_f(x) = 1/3\cdot x^3+c$, where c is a constant to be computed later. Let be

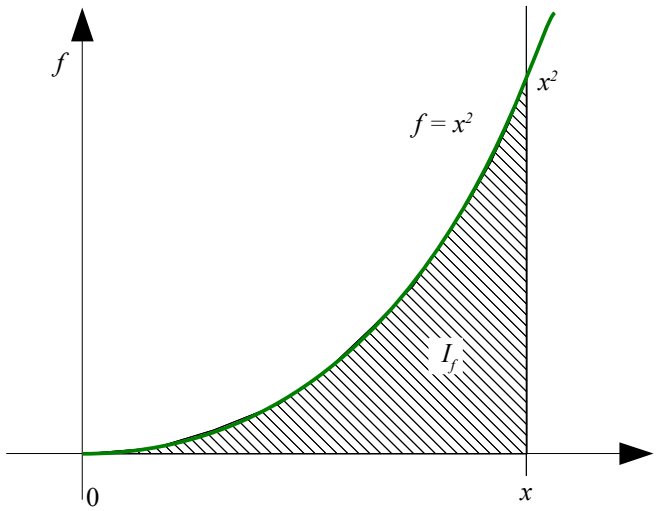
$$\begin{aligned} x\cdot\Delta x\leq\ & /\ \cdot(x+\Delta x)+c-\ /\ \cdot x\ -c \\ =\ & /\ \cdot(x+\ \cdot x\cdot\Delta x+\ \cdot x\cdot\Delta\ x+\Delta\ x)-\ /\ \cdot x \\ =\ & x\cdot\Delta x+x\cdot\Delta\ x+\Delta\ x/ \\ =\ & (x\ +x\cdot\Delta x+\Delta\ x/\ )\cdot\Delta x \\ \leq\ & (x+\Delta x)\cdot\Delta x=(x\ +\ \cdot x\cdot\Delta x+\Delta\ x)\cdot\Delta x. \end{aligned}$$

Simplification yields

$$\leq\ x+\Delta x/\quad\leq\quad\cdot x+\Delta x$$

and shows easily the correctness of this inequalities. This inequalities hold for any positive x and any positive Δx. Thus

$I_f(x) = 1/3\cdot x^3+c$.



The constant c can be found by determining I_f at a position where this area is known. For example if $x = 0$ and $I_f(0) = 0$, then we find for c

$I_f(0) = 0 = 1/3\cdot x^3+c = 1/3\cdot 0^3+c = 0+c = c$.

In this case $c = 0$. For the problem considered the '*definite integral*' (to be defined later) of the parabola, that is the area in the interval $[0,x]$ is

$I_f(x) = 1/3\cdot x^3$.

We can take another lower value, where the *definite integral* is assumed to be 0, for example $u = 2$, then

$I_f(2) = 0 = 1/3\cdot 2^3+c = 8/3+c$,

or

c = –8/3.

thus the definite integral in [2,x] is

$I_f(x) = 1/3 \cdot x^3 - 8/3$.

4.1.4 Exercises

1. Show: if $f(x) = a \cdot x^2$, then $I_f(x) = a/3 \cdot x^3 + c$.
2. Show: if $f(x) = a \cdot x^2 + b \cdot x$, then $I_f(x) = a/3 \cdot x^3 + b/2 \cdot x^2 + c$.
3. Show: if $f(x) = a \cdot x^2 + b \cdot x + c$, then $I_f(x) = a/3 \cdot x^3 + b/2 \cdot x^2 + c \cdot x + d$.
4. Show: if $f(x) = x^3$, then $I_f(x) = 1/4 \cdot x^4 + c$.
5. Which law do you assume?
6. Solve 1. to 3. also with help of rule of linearity and addition from next section .

5 Definition and properties of primitive

Let f be a function. Let I_f be another function with the properties:

If in an interval $[u,v]$ where f is monotonously ascending and each x and each $y = x+\Delta x$ from this interval where $u \le x < y \le v$, thus $\Delta x = y-x > 0$ holds

$$f(x)\cdot\Delta x \le I_f(x+\Delta x) - I_f(x) \le f(x+\Delta x)\cdot\Delta x. \tag{4}$$

If in an interval $[u,v]$ where f is monotonously descending and each x and each $y = x+\Delta x$ from this interval where $u \le x < y \le v$, thus $\Delta x = y-x > 0$ holds

$$f(x)\cdot\Delta x \ge I_f(x+\Delta x) - I_f(x) \ge f(x+\Delta x)\cdot\Delta x. \tag{5}$$

This function I_f is uniquely determined by f, besides a constant; we call I_f a *primitive* of f. Its properties are the following. If (4) or (5) holds, then follows

1. Linearity: $I_{a\cdot f} = a\cdot I_f$.
   $$a\cdot f(x)\cdot\Delta x \le a\cdot I_f(x+\Delta x) - a\cdot I_f(x) \le a\cdot f(x+\Delta x)\cdot\Delta x.$$
   Analogous for (5). Follows from multiplication by a. If $a < 0$, then an ascending function becomes descending, and vice versa, and the inequalities sign have to be altered.
2. Additivity: $I_{f+g} = I_f + I_g$
   $$(f(x)+g(x))\cdot\Delta x \le I_f(x+\Delta x) - I_f(x) + I_g(x+\Delta x) - I_g(x) \le (f(x+\Delta x)+g(x+\Delta x))\cdot\Delta x.$$
   Follows from addition of the inequalities, when both equation are monotonously ascending or descending. We write also $I_{f+g} = I_f + I_g$:
   $$(f(x)+g(x))\cdot\Delta x \le I_{f+g}(x+\Delta x) - I_{f+g}(x) \le (f(x+\Delta x)+g(x+\Delta x))\cdot\Delta x.$$
3. Translocation invariance: $I_{f(x+c)}(x+c) = I_f(x)$.
   $$f(x+c)\cdot\Delta x \le I_f(x+c+\Delta x) - I_f(x+c) \le f(x+c+\Delta x)\cdot\Delta x.$$
   This follows since shifting in x-direction concerns both functions, f and I_f; formally it can be shown by substitution of $z = x+c$; and since $y+c = x+c+\Delta x$ is also $\Delta z = y+c-(x+c) = y-x = \Delta x$, thus
   $$f(z)\cdot\Delta z \le I_f(z+\Delta z) - I_f(z) \le f(z+\Delta z)\cdot\Delta z.$$
   For positive c both functions are shifted to the left, for negative c to the right.

5.1 Notation

A function I_f for a function f, meeting (4, 5), is called *primitive* of f and written in standard mathematics as

$$\int_u^v f(x,y)\,dx = I_{f(x,y)}(v) - I_{f(x,y)}(u), \quad \text{fundamental theorem of Calculus} \tag{6}$$

The left expression is as symbolic notation for a *definite integral*, that displays the function f with its parameter (x and y) as well as the parameter (here x in the *differential* dx), 'over which is to be integrated'. This symbolic notation is called *definite integral* of f in interval $[u,v]$ and yields a number, meaning the area beneath the curve f in interval $[u,v]$ (as long as function f is positive). The notion *integral* is derived from another word for (general) summation, which was the original intention of this method. Other words found in literature (that can be proved to be equivalent) are *definite integral*, *primitive*, *antiderivative*, and *integral* (the latter of which is not universally accepted). Fomula (6), stating a relationship between the *definite integral* and the *primitive*, is also called *fundamental theorem of calculus.* We prefer the notion *primitive*, although others may be used sometimes.

If the lower bound u is fixed, then follows from (6)

$$\int_u^v f(x,y)\,dx = I_f(v) + c\,,$$

with an adequate constant $c = -I_{f(x,y)}(u)$. In this presentation the integral can be regarded as continuos function of its upper bound v (or analogous of its lower bound). If a primitive is to be used to determine the integral (that is the area beneath a curve), then there is a constant to be added, that depends on the lower bound of the integral.

$$\int_u^v f(x)\,dx = I_f(v) - I_f(u)\,,\quad \text{definite integral (a number).}$$

$$\int f(x)\,dx = I_f\,,\quad \text{primitive = indefinite integral (set of functions).}$$

Notation: If f is a function, then I_f is its primitive. If $f(x) = 2\cdot x$, we also write $I_{2\cdot x}(x) = I_f(x)$, where the parameter x in the index is independent of the parameter x in the function; the function $f(x) = 2\cdot x$ is defined implicitly. A function with a name, like sine (sin) or logarithm (log) is written as $I_{\sin}(x)$ or $I_{\log}(x)$ with no parameter x in the index. The meaning should always be clear from the given context.

When function become too intricate, we use either the classical notation or write with the same meaning

$$\int_u^v f(x)\,dx = I_f(x=v) - I_f(x=u) = I\left[\,f(x)\right]_{x=u}^{x=v},$$

This includes interval borders with definite integrals. Here we can specify the integration variable (in the example x), if the function holds several variables. In the example

$$I\left[\,(z-x)^k\right]_{x=u}^{x=v},$$

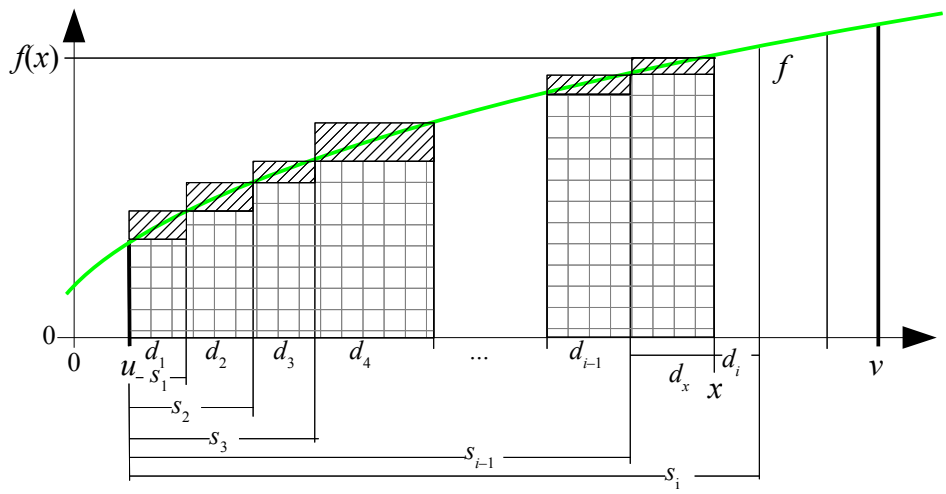
integration is to be done over x and not over z.

5.2 Uniqueness of the primitive

We prove here and in the next sections some fundamental properties of primitives, which are of theoretical interest. They can be skipped, if the reader is only interested in practical questions.

To prove uniqueness of the primitive let as assume there is an *areal function* I_f, determining the area beneath the curve f, that fulfils

$$f(x)\cdot\Delta x \le I_f(x+\Delta x) - I_f(x) \le f(x+\Delta x)\cdot\Delta x\,. \tag{7}$$



We will show that this function is uniquely determined (besides a constant addend) and it has the

same value as the *areal function* of f (which we will call A_f in this section). To do this we compute the areas of the strips that encloses the areal function and show that both, any function I_f meeting conditions (7) and the areal function A_f are identical.

Let f be positive and monotonously ascending in interval $[u,v]$; let the areal function A_f be zero in u, i.e. $A_f(u) = 0$. The area beneath the function is partitioned in stripes of width d_i, where each width is bounded by d: $d_i \le d$. Partitioning of intervals be arbitrary, however fixed. Area of these rectangles is described by two functions F und F^+; F stands for the area of the rectangles beneath function f and F^+ for the area of the rectangles above function f.

In detail we define $s_i = s_{i-1}+d_i$, $s_0 = 0$, where s_i is the width of the first i stripes, since $s_0 = 0$, $s_1 = d_1$, $s_2 = d_1+d_2$ etc. Thus we get for F and F^+

$$F(x) = f(u)\cdot d_1+f(u+s_1)\cdot d_2+f(u+s_2)\cdot d_3 + \ldots + f(u+s_{k-1})\cdot d_k + \ldots + f(u+s_{i-2})\cdot d_{i-1} + f(u+s_{i-1})\cdot d_x,$$

$$F^+(x) = f(u+s_1)\cdot d_1+f(u+s_2)\cdot d_2+f(u+s_3)\cdot d_3 + \ldots + f(u+s_k)\cdot d_k + \ldots + f(u+s_{i-1})\cdot d_{i-1} + f(x)\cdot d_x.$$

Here $x \in [s_{i-1},s_i]$ and $d_x = x-u-s_{i-1}$, and thus $d_x \le d_i \le d$. Then for each positive d the areal function A_f is larger than the sum of the strips beneath the curve f, i.e. $F_d(x) \le A_f(x)$; but for each positive d the areal function A_f is smaller than the sum of the strips above the curve f, i.e. $A_f(x) \le F_d^+(x)$. For the areal function A_f follows from definition: $f(u+s_{k-1})\cdot d_k \le A_f(u+s_k)-A_f(u+s_{k-1}) \le f(u+s_k)\cdot d_k$, and thus

$$F(x) \le A_f(x) \le F^+(x). \tag{8}$$

Now we compute the difference of $F^+(x)$ and $F(x)$. From the diagram follows

$$F^+(x) - F(x) \le (f(x) - f(u))\cdot d. \tag{9}$$

More formally this can be shown algebraically by summation of $f(u+s_{i+1})-f(u+s_i)$; since they are positive differences we gets

$$F^+(x) - F(x) =$$

$$f(u+s_1)\cdot d_1 + f(u+s_2)\cdot d_2 + f(u+s_3)\cdot d_3 + \ldots + f(u+s_i)\cdot d_{i-1} + f(x)\cdot d_x -$$
$$(f(u)\cdot d_1 + f(u+s_1)\cdot d_2 + f(u+s_2)\cdot d_3 + \ldots + f(u+s_{i-2})\cdot d_{i-1} + f(u+s_{i-1})\cdot d_x)=$$

$$f(u+s_1)\cdot d_1 - f(u)\cdot d_1 + f(u+s_2)\cdot d_2 - f(u+s_1)\cdot d_2 + f(u+s_3)\cdot d_3 - f(u+s_2)\cdot d_3 + \ldots + f(x)\cdot d_x - f(u+s_{i-1})\cdot d_x =$$

$$(f(u+s_1)-f(u))\cdot d_1 + (f(u+s_2) - f(u+s_1))\cdot d_2 + (f(u+s_3) - f(u+s_2))\cdot d_3 + \ldots + (f(x) - f(u+s_{i-1}))\cdot d_x \le$$

$$(f(u+s_1)-f(u))\cdot d + (f(u+s_2) - f(u+s_1))\cdot d + (f(u+s_3) - f(u+s_2))\cdot d + \ldots + (f(x) - f(u+s_{i-1}))\cdot d =$$

$$= f(x)\cdot d - f(u)\cdot d = (f(x) - f(u))\cdot d.$$

Now we derive analogous inequalities for the functions I_f. According to definition (7) for any function I_f and for any strip at any point x – thus also at the points $x = u+k\cdot d$ – follows with $\Delta x = d$

$$\begin{aligned}
&f(u)\cdot d_1 \le I_f(u+s_1) - I_f(u) \le f(u+d_1)\cdot d_1,\\
&f(u+s_1)\cdot d_2 \le I_f(u+s_2) - I_f(u+s_1) \le f(u+s_2)\cdot d_2,\\
&f(u+s_2)\cdot d_3 \le I_f(u+s_3) - I_f(u+s_2) \le f(u+s_3)\cdot d_3,\\
&\ldots\\
&f(u+s_{i-2})\cdot d_{i-1} \le I_f(u+s_{i-1}) - I_f(u+s_{i-2}) \le f(u+s_{i-1})\cdot d_{i-1},\\
&f(u+s_{i-1})\cdot d_x \le I_f(x) - I_f(u+s_{i-1}) \le f(x)\cdot d_x
\end{aligned} \tag{10}$$

We can sum those inequalities and get other valid inequalities.

$$F(x) \le I_f(x) - I_f(u) \le F^+(x). \tag{11}$$

From these inequalities follows

$$-F^{+}(x) \le -(I_f(x) - I_f(u)) \le -F(x),$$

and summing with (8) yields

$$-(F^{+}(x) - F(x)) \le A_f(x) - (I_f(x) - I_f(u)) \le F^{+}(x) - F(x).$$

Finally follows from (9)

$$|A_f(x) - (I_f(x) - I_f(u))| \le F^{+}(x) - F(x) \le (f(x) - f(u))\cdot d \tag{12}$$

for each positive d. This means that deviation between areal function $A_f(x)$ and $I_f(x) - I_f(u)$ is less than deviation between $F_d^{+}(x)$–$F_d(x)$.

With this estimation follows immediately that the areal function A_f and any of the functions I_f that fulfil (7) are identical.

$$A_f(x) = I_f(v) - I_f(x).$$

Otherwise we could write

$$A_f(x) = I_f(x) - I_f(u) + h, \tag{13}$$

where the value of $h = A_f(x) - (I_f(x) - I_f(u))$ determines a deviation of these functions at point v. If h is not zero, i.e. $|h| > 0$, and if $f(v)$ and $f(u)$ are different numbers, we can select a positive

$$d < |h|/(f(v) - f(u)),$$

for example

$$d = |h|/(f(v) - f(u))/2,$$

for which inequalities (7) hold as well, since they hold for <u>each</u> positive $d = \Delta x$, where maximum deviation is according to (9)

$$0 \le |A_f(x) - (I_f(x) - I_f(u))| = |h| \le F^{+}(x) - F(x) \le (f(x) - f(u))\cdot d = |h|/2.$$

From $|h| \le |h|/2$ follows that h = 0 as unique solution. Thus deviation of $A_f(v)$ and $I_f(v) - I_f(u)$ is zero.

We can prove the same for functions that are descending in an interval, as well as for function with finite variability.

This construction is only made to prove the uniqueness. It never has to be realized with concrete fucntions. Since integration is usually introduced by summation of those strips beneath the curve we get here a method to prove such summation formulae.

Let be $v = x$ a variable, then we defined the primitive as

$$A_f(x) = I_f(x) + c,$$

with $c = -I_f(u)$ (since then $A_f(u) = 0 = I_f(u)+c = I_f(u)-I_f(u)$). Areal function and primitive – as defined in (4) – coincide, besides a constant c.

This construction is only made to prove the uniqueness. It never has to be realized. Since integration is usually introduced by summation of those strips beneath the curve we get here a method to prove such summation formulae. See exercise 11.2.3.

5.3 Continuity of the primitive

If function f is limited (that means each value of f in the interval is less than a maximum number M), then I_f is continuos, since from (4) follows that any ascent of I_f can be limited by any positive h, by setting

$$\Delta x < \frac{h}{f(x+\Delta x)}$$

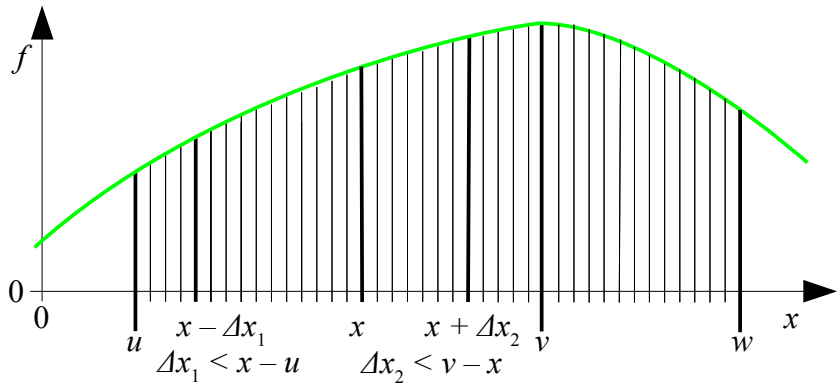
adequately. Then follows

$$I_f(x+\Delta x) - I_f(x) \leq f(x+\Delta x)\cdot\Delta x < h.$$

Analogous holds for (5).

5.4 Primitive for non-monotonous functions

A primitive for a function that is not monotonous can usually be found by partitioning the range of definition into intervals in which the function is monotonous. Then for each interval the primitive is computed and thus can be constructed a primitive for all values.



If a function alters its direction more and more often, then the sum of all alterations may increase. The sum of all $|f(v_i) - f(u_i)|$ for intervals $[u_i, v_i]$ in which f is monotonous is called 'variability' of function f in the total range. Since the deviation depends on this variability, it must be limited. If this cannot be guaranteed, then the function is not integrable.

An example of such a function with unlimited variability in a finite interval is $\cos(1/x)$, where with falling x the number of periods are growing more and more. This function is not integrable in our theory, as well not as Riemann integral. Even if there exists a primitive to such a function, e.g. for the function

$$f(x) = \cdot x \cdot \sin\frac{}{x} - \cos\frac{}{x}$$

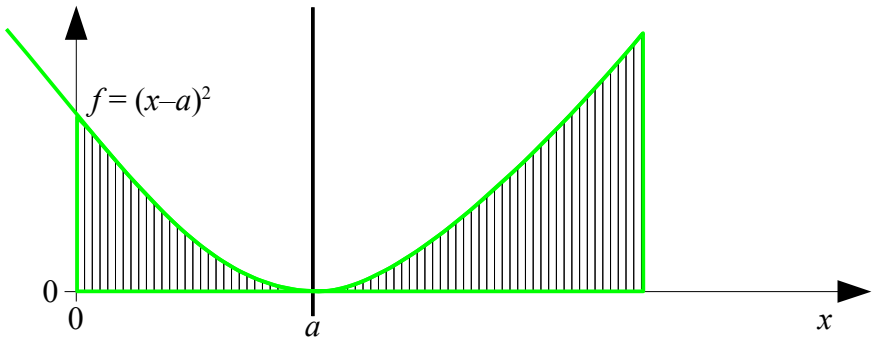
the primitive function is

$$I_f(x) = x \cdot \sin\frac{}{x},$$

this primitive cannot be evaluated for $x = 0$.

5.4.1 Example

Integrate $f(x) = (x–a)^2$ for positive x; its primitive is $I_f(x) = 1/3\cdot(x–a)^3$. The function f decreases in interval $[0, a]$ and increases then infinitely.



For the first interval $[0,a]$ we show

$$f(x)\cdot\Delta x \geq I_f(x+\Delta x)-I_f(x) \geq f(x+\Delta x)\cdot\Delta x, \quad \text{for } 0 \leq x \leq a \text{ and } 0 < \Delta x < a-x \tag{14}$$

With the function f extended in (14) we get the inequalities

$$(x-a)\cdot\Delta x \geq \;/\;\cdot(x+\Delta x-a) - \;/\;\cdot(x-a) \geq (x+\Delta x-a)\cdot\Delta x, \quad \text{for } \Delta x < a-x$$

or

$$(x-a)\cdot\Delta x \geq (x-a)\cdot\Delta x+(x-a)\cdot\Delta\;x+\;/\;\cdot\Delta\;x \geq ((x-a)+\;\cdot(x-a)\cdot\Delta x+\Delta\;x)\cdot\Delta x,$$

or

$$\geq (x-a)\cdot\Delta\;x+\;/\;\cdot\Delta\;x \geq \;\cdot(x-a)\cdot\Delta\;x+\Delta\;x,$$

or

$$\geq (x-a)+\;/\;\cdot\Delta x \geq \;\cdot(x-a)+\Delta x.$$

The first inequality follows from $1/3\cdot\Delta x < \Delta x < a-x$. After reordering follows from the second inequality

$$a-x \geq \;/\;\cdot\Delta x,$$

this holds $\Delta x < a-x$ obviously.

In the second interval is $x \geq a$, so we show inequalities (4). Since function f is monotonously ascending we must prove

$$f(x)\cdot\Delta x \leq I_f(x+\Delta x)-I_f(x) \leq f(x+\Delta x)\cdot\Delta x, \quad \text{for } a \leq x \text{ and } 0 < \Delta x.$$

We have to show that

$$(x-a)\cdot\Delta x \leq \;/\;\cdot(x-a+\Delta x) - (x-a) \leq (x-a+\Delta x)\cdot\Delta x,$$

or

$$(x-a)\cdot\Delta x \leq (x-a)\cdot\Delta x+(x-a)\cdot\Delta\;x+\;/\;\cdot\Delta\;x \leq ((x-a)+\;\cdot(x-a)\cdot\Delta x+\Delta\;x)\cdot\Delta x,$$

or

$$\leq (x-a)\cdot\Delta\;x+\;/\;\cdot\Delta\;x \leq \;\cdot(x-a)\cdot\Delta\;x+\Delta\;x,$$

or

$$\leq x-a+\;/\;\cdot\Delta x \leq \;\cdot(x-a)+\Delta x.$$

From the first inequality follows

$$a \leq x+\;/\;\cdot\Delta x,$$

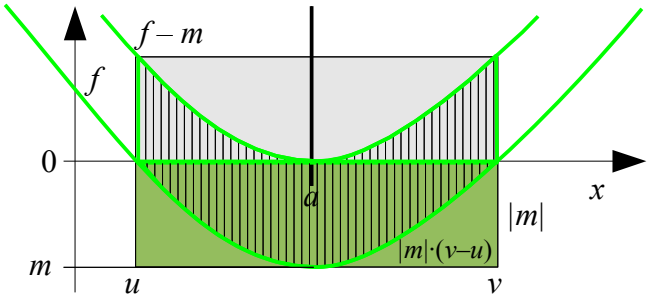
this holds because of $a \leq x$. From the second inequality follows after reordering

$a \leq x + \quad / \quad \cdot \Delta x.$

In both cases the primitive of f is $I_f = 1/3 \cdot (x-a)^3$; thus the result is that the primitive of $f(x) = (x-a)^2$ is the function $I_f = 1/3 \cdot (x-a)^3$ for all positive values, so that we have found the primitive of $(x-a)^2$.

5.5 Negative function values

We will now treat the case of negative function values. In an interval $[u, v]$ the function f will always be negative, where $m = \min f(x)$ is the smallest value of the function in this interval. Then the function $f(x)-m$ is always positive and can be integrated with our method in the interval $[u, v]$; from this result we can subtract $(v-u) \cdot |m|$.



The result is the area between the x-axis and the curve, however with a negative value. This is because the sum of the strips $\Delta x \cdot f(x)$ beneath the x-Achse, have still positive $\Delta x > 0$, however negative $f(x) < 0$; the product $\Delta x \cdot f(x) < 0$ is therefore negative, thus also the 'size' of the area. Negative values of area in sum intervals can be cancelled by positive values in other.

From this follows if the sign of a function changes then the zero points of the function are to be estimated and within those intervals integration is to be done independently. The formal value of an integral may become positive or negative.

5.5.1 Example

We integrate the function $f(x) = x^2-4 \cdot x+3$; it is negative in the interval [1,3] (see last diagram for principal behaviour of that function). The indefinite integral is $I_f(x) = x^3/3-2 \cdot x^2+3 \cdot x$. The definite integral in the interval [0,1] is 4/3. The definite integral in the interval [1,3] is

$I_f(3) - I_f(1) = 27/3 - 2 \cdot 3^2+3 \cdot 3 - (1^3/3-2 \cdot 1^2+3 \cdot 1) = 0 - 4/3 = -4/3.$

Thus the integral in interval [0,3] is 0, although the function is not zero. If the geometric area beneath curve f and x-axis in this interval is to be computed, the absolute values are to be summed, i.e. 4/3 + 4/3 = 8/3. Alternatively the function can be moved by the minimal value -1 and then integrated. Then the area of the corresponding rectangle is to be subtracted again. Then follows with $g(x) = f(x)+1 = x^2-4 \cdot x+4$

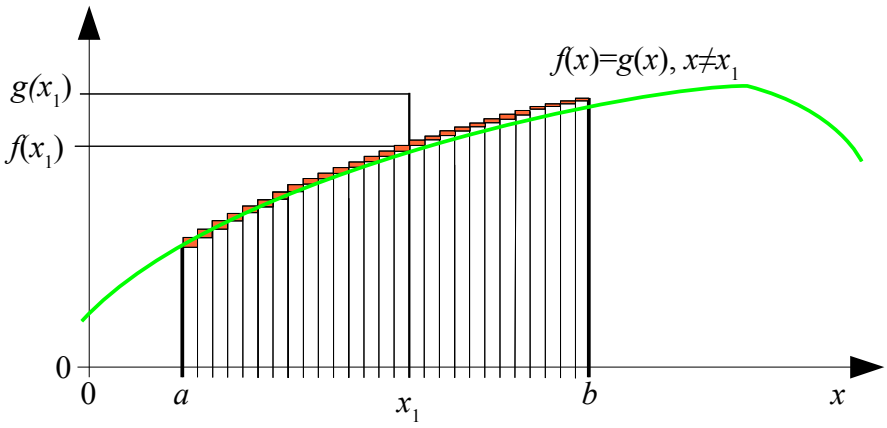
$I_g(3) - I_g(0) = 27/3 - 2 \cdot 3^2+4 \cdot 3 - 0 = 3.$

The rectangle has the size $|-1| \cdot (3-0)=3$, so that the definite integral of f in the interval [0,3] has the value 0.

5.6 Zero integrals

For a continuos function f (that is not necessarily monotonous) there is in general a unique primitive I_f. But there can be several distinct functions g, h with the same primitive $I_f = I_g = I_h$. Thus integration is not uniquely invertible.

For a function $f(x)$ and another function g, that differs from f only at one point $x = 1$, the integral as the area beneath the curves f and g is the same in both cases, $I_f = I_g$.



Let $t = g–f$ be the difference of that two functions. $t(x)$ is zero at all points but $x = 1$

$$t(x)=\begin{cases} , & \text{falls } x = \\ , & \text{sonst} \end{cases} .$$

From additivity follows

$$I_t = I_g - I_f = 0,$$

so that $I_t = 0$. This holds for any finite number of distinct numbers x_i with $t(x_i) > 0$, and $t(x) = 0$ for all other x of this function, the integral of t is 0, and if t is added to a function f, so that with $g = f+t$ follows $I_g = I_f$. From this follows that to a primitive I_f of f arbitrarily many distinct functions g exists with the same integral $I_g = I_f$.

Although this is no problem for integration, it is a problem for the inverted function to a primitive, which will be introduced in the next section. While the most functions also with finite number of 'steps' can be integrated, their inverted function cannot be determined uniquely.

6 Inverting Integration: Differentiation

We look for a function f of a function I_f, so that (4) is fulfilled. The process is called *Differentiation* and is similar important as integration.

6.1 Definition of Derivative

Given a function g we search for a function D_g, so that inequalities (4) and (17) hold for $g = I_{Dg}$. For a monotonously ascending function g we get

$$\begin{aligned} D_g(x)\cdot\Delta x \le g(x+\Delta x)-g(x) \le D_g(x+\Delta x)\cdot\Delta x\,, \\ D_g(x-\Delta x)\cdot\Delta x \le g(x)-g(x-\Delta x) \le D_g(x)\cdot\Delta x\,. \end{aligned} \tag{15}$$

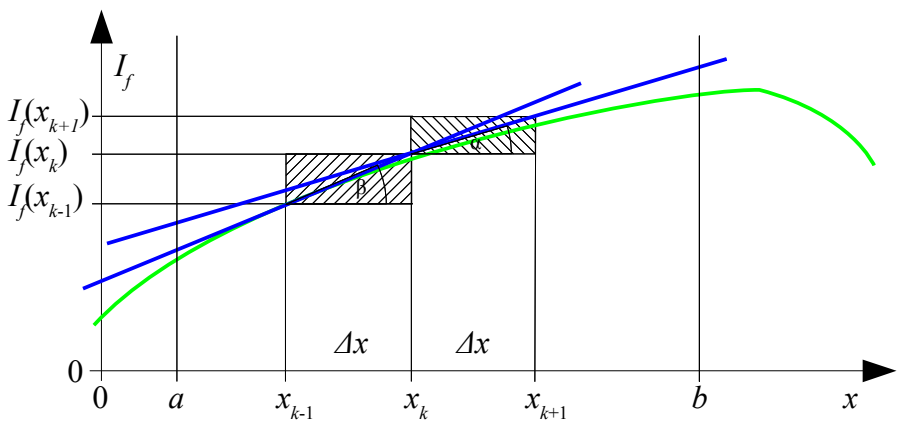
This function D_g is called the *derivation* of g, if at least $g = I_{Dg}$, but this must not hold for only 'one' side.

There are many problems with the definition of the derivative, so that differentiation is more complicated than integration. Since the inverse function of a primitive is not unique, one of those inverse functions is to be selected. Usually the continuos function is choosen. If there is a continuos function D_g with property (15), then D_g is called the derivative of g. This process is called differentiation. Since D_g is continuos g is called continuously *differentiable* ('continuos' is therefore D_g while g *has to be* continuos anyway).

From this definition follows

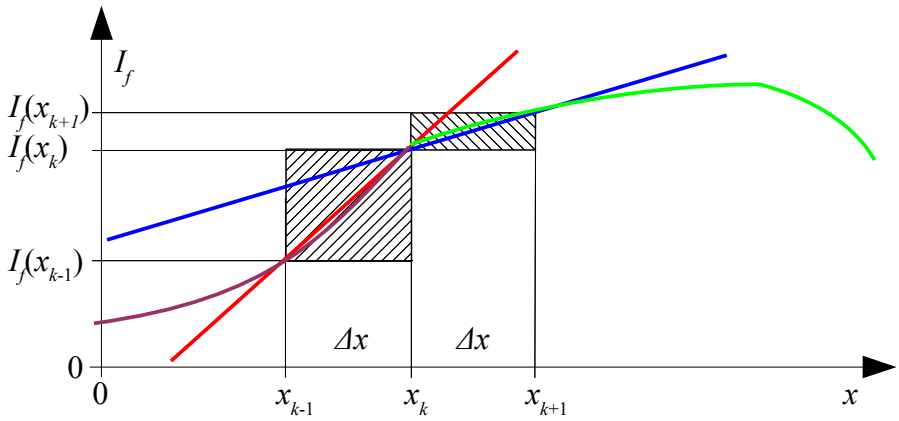
$$g = I_{D_g}\,,\; f = D_{I_f}\,,$$

provided the corresponding function exist.



Let there be a derivative D_g of a function g, so that properties (15) hold in each point of the interval from both sides. To understand the meaning of the derivative the primitive $I = I_f$ of a function f is plotted. The ratio of the difference of the values for the primitive to the interval width can be interpreted as ascent of the tangent to the primitive in point x_k. By ascent (sometimes called gradient) we mean here the tangent of the angle of the secant of the curve I_f in the points x_k and x_{k+1} with the horizontal, i.e. tan α, tan β, etc.

From the diagram follows that the angles α and β of the secants in x_k are almost the same, and become identical when Δx becomes smaller and smaller. But that must not hold in all cases, as is shown in the next diagram.



In point x_k the curve I_f is bend, so that the ascent of the line (for small Δx) differs always. In this case the derivative does not exist. Thus we define that a derivative D_g is defined only if

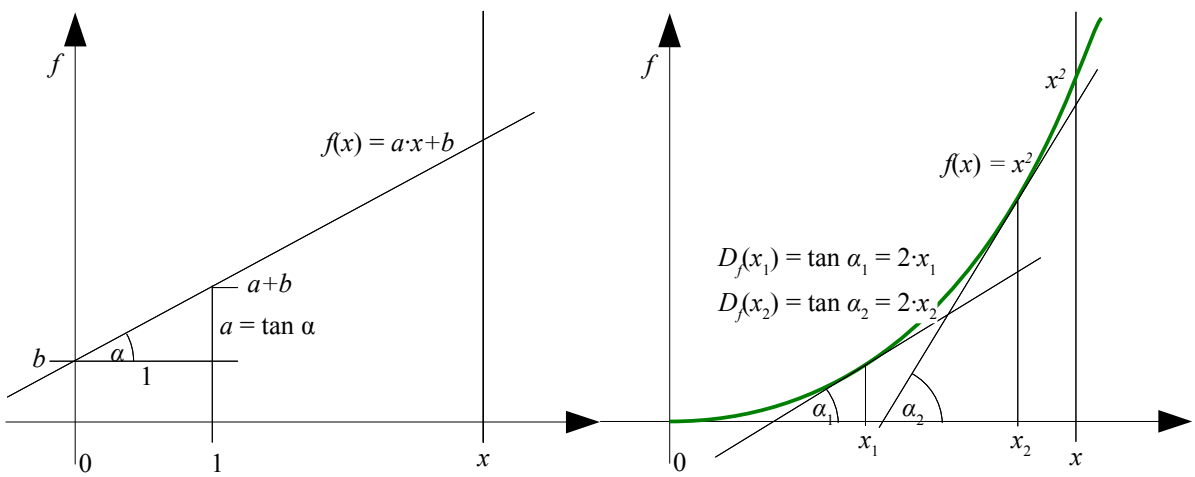
$$\begin{aligned} D_g(x)\cdot\Delta x \le g(x+\Delta x)-g(x) \le D_g(x+\Delta x)\cdot\Delta x\,, \\ D_g(x-\Delta x)\cdot\Delta x \le g(x)-g(x-\Delta x) \le D_g(x)\cdot\Delta x\,. \end{aligned} \tag{16}$$

for a continuos function D_g for each x in an interval $[a,b]$ and each $\Delta x > 0$. Function g is then called continuously differentiable in the interval $[a,b]$.

6.1.1 Example

We show the plausibility of the definition by differentiating a straight line. The ascent of the straight line $f(x) = a\cdot x+b$ must be the derivative of this function. The ascent is for two points 0 and 1:

$$\frac{f(\ )-f(\ )}{-}=\frac{(a+b)-(\ +b)}{}=a\,.$$



To find $D_f(x) = a$ we must show that definition (15) holds, or that holds

$$D_f(x)\cdot\Delta x = a\cdot\Delta x \le a\cdot(x+\Delta x)+b-a\cdot x-b = a\cdot\Delta x \le D_f(x+\Delta x)\cdot\Delta x = a\cdot\Delta x\,,$$

Obviously, these inequalities are always correct, since all terms are identical. Thus, for this simple case our interpretation of the derivative as ascent of a function in a point is correct.

Another example is given to demonstrate the usefulness of this definition. Let be $f(x) = x^2$. Then we must show that $D_x(x) = \ \cdot x$, since $I_{2\cdot x}(x) = x^2$.

$$\begin{aligned} D_x(x)\cdot\Delta x = \ \cdot x\cdot\Delta x \le (x+\Delta x)\ -x\ \le\ \cdot(x+\Delta x)\cdot\Delta x = D_x(x+\Delta x)\cdot\Delta x\,, \\ D_x(x-\Delta x)\cdot\Delta x = \ \cdot(x-\Delta x)\cdot\Delta x \le x\ -(x-\Delta x)\ \le\ \cdot x\cdot\Delta x = D_x(x)\cdot\Delta x\,; \end{aligned}$$

or

$$\cdot x\cdot\Delta x\leq\ \cdot x\cdot\Delta x+\Delta\ x\leq\ \cdot x\cdot\Delta x+\ \cdot\Delta\ x,$$
$$\cdot x\cdot\Delta x-\ \cdot\Delta\ x\leq\ \cdot x\cdot\Delta x-\Delta\ x\leq\ \cdot x\cdot\Delta x;$$

or

$$\leq\Delta\ x\leq\ \cdot\Delta\ x,$$
$$-\ \cdot\Delta\ x\leq-\Delta\ x\leq\ .$$

Since the last inequalities hold, the first ones hold. Thus follows

$$D_x(x)=\ \cdot x.$$

The derivative D_f of f is a function itself, that is defined at all points where f is defined and continuously differentiable. It is uniquely defined by inequalities (16) and yields the ascent of the function f in a point x. We will later present the derivatives for several functions, and also give some rules to compute such derivatives. Inequalities (16) are a definition when an derivative is defined and how to check, whether D_f is the derivative to a function f.

6.2 Properties of derivatives

For many functions there exist no derivatives, however they are integrable. All functions with steps or sharp bends are not differentiable, although there are some others that are not differentiable as well. Pragmatically one can define:

If there is for a function g in some or all points no unique ascent of the tangent, then the function is not differentiable.

If the derivative of a function is continuos, then the function is called continuously differentiable; we will consider only continuously differentiable functions.

Derivatives can be defined from two sides differently; this will not be done in this paper.

At the borders of an interval $[a,b]$ the derivatives must be defined particularly, e.g. from one side only or with the function beyond the interval borders.

We will always assume that functions are differentiable and satisfy definition (16), so that for a primitive g the derivative D_g is uniquely defined.

6.3 Uniqueness and Meaning of Derivative

When inequalities (16) hold and the derivative D_f is continuos then D_f is uniquely determined. This can easily be proved since the difference between two derivatives of f must be less than any number. Formal prove is given in next section.

Expression

$$q_f(x,\Delta x)=\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

is called differential quotient. We assume that it is a function of Δx, when x is fixed. For $\Delta x = 0$ it becomes an indifferent expression 0/0, that can take any value. We define $q_f(x,\Delta x)$ as

$$q_f(x,\ )=D_f(x)$$

and get a function q_f, that meats condition (16),

$$D_f(x) \le q_f(x, \Delta x) \le D_f(x+\Delta x),$$
$$D_f(x-\Delta x) \le q_f(x, \Delta x) \le D_f(x),$$

also, when Δx becomes zero. Geometric meaning of $q_f(x,0)$ is ascent of tangent of function f at point x. As ascent of tangent of function f in point x we define ascents of all secants close to x, where we can show that all these are equal to $D_f(x)$. That will be shown in the section after the next one.

Altogether we find that any continuos derivative D_f, meeting (16), is uniquely defined and its value is the ascent of the tangent of the function f.

6.3.1 Uniqueness

Let be D_f and E_f two continuos function, meeting the inequalities

$$D_f(x)\cdot\Delta x \le f(x+\Delta x) - f(x) \le D_f(x+\Delta x)\cdot\Delta x,$$
$$D_f(x-\Delta x)\cdot\Delta x \le f(x) - f(x-\Delta x) \le D_f(x)\cdot\Delta x,$$

and

$$E_f(x)\cdot\Delta x \le f(x+\Delta x) - f(x) \le E_f(x+\Delta x)\cdot\Delta x,$$
$$E_f(x-\Delta x)\cdot\Delta x \le f(x) - f(x-\Delta x) \le E_f(x)\cdot\Delta x,$$

for each positive Δx and each x in interval $[u,v]$. If these functions differ at z

$$h = E_f(z) - D_f(z)$$

where h is positive: $E_f(z) > D_f(z)$. From continuity of D_f follows there is a positive d, so that $D_f(z)$ and $D_f(z+d)$ differ less than $h/2$, where because of inequalities

$$D_f(z) \le D_f(z+d) \le D_f(z)+h/2;$$

Then

$$D_f(z) \le \frac{f(z+d)-f(z)}{d} \le D_f(z+d) \le D_f(z)+h/$$

Since also $E_f(z)$ is bounded by this quotient follows

$$D_f(z) \le D_f(z)+h = E_f(z) \le \frac{f(z+d)-f(z)}{d} \le D_f(z)+h/ \quad .$$

From $0 \le h \le h/2$ follows $h = 0$. Thus D_f and E_f cannot be distinct.

6.3.2 Meaning of the derivative

Ascent of a secant of function f at points y and z ($y < z$) is defined as

$$\frac{f(z)-f(y)}{z-y} = q(y, z-y).$$

Since D_f is continuos, there is a d_x for each x and each positive s, so that

$$D_f(x-d_x) \le D_f(x+d_x) \le D_f(x-d_x)+s.$$

Let be given two pairs of points $y_1,z_1 \in [x-d_x,x+d_x]$ and $y_2,z_2 \in [x-d_x,x+d_x]$, where $y_1<z_1$ und $y_2<z_2$, and let be the ascent of the secant through y_1,z_1 less than that through y_2,z_2, then follows

$$D_f(x-d_x) \le D_f(y) \le \frac{f(z)-f(y)}{z-y} \le \frac{f(z)-f(y)}{z-y} \le D_f(z) \le D_f(x+d_x) \le D_f(x-d_x)+s.$$

Then the ascents of the secants close to x differ less than s from each other. Since D_f is monotonous follows also

$$D_f(x-d_x) \le D_f(x) \le D_f(x+d_x) \le D_f(x-d_x)+s\,,$$

thus the ascents of the secants close to x differ less than s from $D_f(x)$.

6.4 Relationship to standard mathematics

Standard mathematics writes for a derivative D_f of a function f mostly f' or more precisely

$$D_f = f' = \frac{df}{dx} = \frac{d}{dx} f\,.$$

Those df are sometimes called differentials, although they are not sober defined. Interpretation of this notation is that the derivative is the ratio of two differential values – df for $f(x+\Delta x)-f(x)$, and dx for Δx – when they decrease to 0. We proved in section 6.3.2 without limits by defining function D_f with inequalities (16) and get uniquely a function D_f giving the ascent of tangent of f.

Multiple differentiation is written as $D_f^{(2)}(x)$ for derivative of $D_f(x)$, $D_f^{(3)}(x)$ for derivative of $D_f^{(2)}(x)$, and $D_f^{(k+1)}(x)$ for derivative of $D_f^{(k)}(x)$. Often one writes $D_f^{(0)}(x)$ for function $f(x)$ and $D_f^{(1)}(x) = D_f(x)$.

6.5 Construction of derivative

To find derivative of D_f of a function f, one can evaluate differential quotient and take its value for very small Δx. For example for polynomials we find simply

$$D_{x^k}(x) \le \frac{(x+\Delta x)^k - x^k}{\Delta x} = k \cdot x^{k-} + \Delta x \cdot (\ldots) \le D_{x^k}(x+\Delta x),$$

where all remaining terms vanish, since they have the factor Δx. For sine function we get

$$D_{\sin}(x) \le \frac{\sin(x+\Delta x) - \sin x}{\Delta x} = \frac{\sin x \cdot (\cos \Delta x - \quad) + \cos x \cdot \sin \Delta x}{\Delta x} \le D_{\sin}(x+\Delta x).$$

Here more complex limit processes are to be considered. For small Δx is cox $\Delta x = 1$, the difference cox Δx–1 therefore zero, while for small Δx is sin Δx approximately Δx, thus the quotient is one. From this follows the derivative $D_{\sin} = \cos$.

All derivatives can and should be verified by inequalities (16). This will be done in the next chapter.

7 Primitive and Derivative of some important functions

In this Section we give the primitives and derivatives to some important functions, using our newly introduced method.

7.1 Polynomials

Polynomials are functions of the form

$$f(x)=\sum_{i=}^{k} a_k \cdot x^k .$$

Since those parameters a_k can be selected very flexible, they are approximating many functions and are very valuable for many problems.

7.1.1 Primitive

Because of linearity and additivity we only have to consider powers like x^k; primitives of general polynomials can then be found by summing primitives of powers with constant factorsa_k. Since powers are unlimited and for positive x monotonously ascending, only inequalities (4) are to be analysed. In general for all powers hold

$$\begin{aligned}
x^k \cdot \Delta x &\leq \frac{}{k+} \cdot \sum_{i=}^{k+} \binom{k+}{i} \cdot x^{k+-i} \cdot \Delta^i x - \frac{}{k+} \cdot x^{k+} = \\
&= \frac{}{k+} \cdot \sum_{i=}^{k+} \binom{k+}{i} \cdot x^{k+-i} \cdot \Delta^i x = \\
&= \sum_{i=}^{k+} \binom{k}{i-} \cdot \frac{}{i} \cdot x^{k+-i} \cdot \Delta^i x = \\
&= \sum_{i=}^{k} \binom{k}{i} \cdot \frac{}{i+} \cdot x^{k-i} \cdot \Delta^{i+} x \leq \\
&\leq \sum_{i=}^{k} \binom{k}{i} \cdot x^{k-i} \cdot \Delta^i x \cdot \Delta x = (x+\Delta x)^k \cdot \Delta x .
\end{aligned}$$

Comparing terms, this proves everything. If x is negative the result follows from symmetry of antisymmetry, if k is odd.

Thus for any polynomial f we get

$$I_{\sum a_k \cdot x^k} = \sum \frac{a_k}{k+} \cdot x^{k+} .$$

Although this is proved only for positive integer exponents k it holds also for any rational (or real) exponent, which will be proved later.

7.1.2 Derivative

For the derivative of a polynomial follows as inversion of integration in the same way

$$D_{\sum a_k \cdot x^k} = \sum a_k \cdot k \cdot x^{k-} .$$

This can be proved directly as well.

$$k\cdot x^{k-}\cdot\Delta x\leq\sum_{i=}^{k}\binom{k}{i}\cdot x^{k-i}\cdot\Delta^{i}x-x^{k}=$$
$$=\sum_{i=}^{k}\binom{k}{i}\cdot x^{k-i}\cdot\Delta^{i}x=$$
$$=k\cdot\sum_{i=}^{k-}\binom{k-}{i}\cdot\frac{}{i+}\cdot x^{k--i}\cdot\Delta^{i}x\cdot\Delta x\leq$$
$$\leq k\cdot\sum_{i=}^{k-}\binom{k-}{i}\cdot x^{k--i}\cdot\Delta^{i}x\cdot\Delta x=k\cdot(x+\Delta x)^{k-}\cdot\Delta x.$$

Comparing the terms proves the inequalities.

We formulate a general power rule

Power rule: Derivative of a power x^k is exponent times x powered by exponent minus 1. (17)

that will be proved later for general exponent k.

7.2 Trigonometric functions

Sine and cosine are trigonometric functions, that will here be integrated via their geometrical interpretation and not over serial development of inversion of their derivatives. This seems to be come across more vividly.

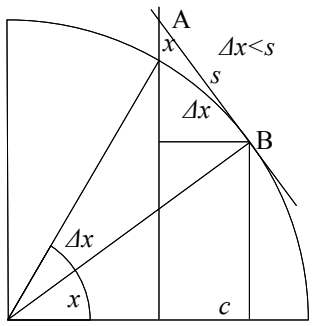
7.2.1 Primitive of sine

Primitive of $\sin x$ is $-\cos x$. We consider only interval $[0,\pi/2]$; others follow from symmetry. We have to prove that

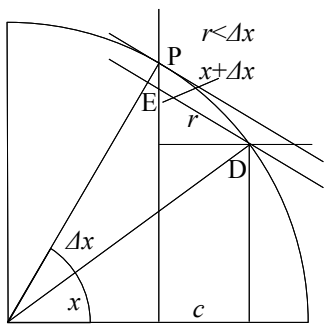
$$\Delta x\cdot\sin x\leq\cos x-\cos(x+\Delta x)\leq\Delta x\cdot\sin(x+\Delta x).$$

Looking at the unit circle, where sine is vertically, cosine horizontally plotted on the axis. In the first graphic the circular arc Δx of angle x to angle $x+\Delta x$ is less than the line s from A to B of the tangent in B; obviously is x the angle at A. Thus follows

$$\Delta x\cdot\sin x\leq s\cdot\sin x=c=\cos x-\cos(x+\Delta x).$$



c=cos(x)-cos(x+Δx)



c=cos(x)-cos(x+Δx)

The second graphic shifts the tangent in P parallel down through point D. The line r from E to D is less than the circular arc of Δx, since r is obviously less than the secant from P to D: $r\leq\Delta x$. The angle in E is obviously $x+\Delta x$. Thus follows

$$\cos x-\cos(x+\Delta x)=c=r\cdot\sin(x+\Delta x)\leq\Delta x\cdot\sin(x+\Delta x).$$

Altogether we have $I_{\sin}(x)=-\cos x$.

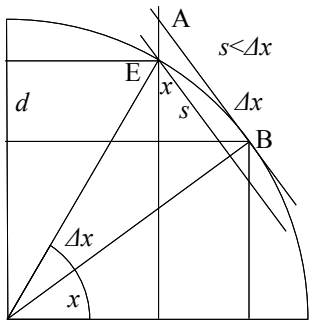
7.2.2 Primitive of cosine

To find the primitive of $\cos x$, we have to show that

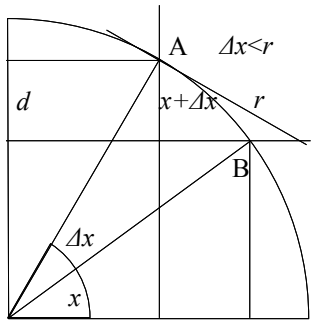
$$\Delta x \cdot \cos x \geq \sin(x+\Delta x) - \sin x \geq \Delta x \cdot \cos(x+\Delta x). \tag{18}$$

Since cosine is monotonously decreasing in interval [0,π] we altered the inequality signs. The first graphic shows the tangent in B being shifted down parallel, cutting E. obviously the angle in E is x. the line s is shorter than the secant from E to B, thus also shorter than the arc Δx. Then follows

$$\sin(x+\Delta x) - \sin x = d = s \cdot \cos x \geq \Delta x \cdot \cos x.$$



d=sin(x+Δx) – sin x



d=sin(x+Δx) – sin x

In the second graphic the angle at tangent in point A with the vertical line is $x+\Delta x$. Obviously r is larger than the arc Δx from A to B, so that follows

$$\sin(x+\Delta x) - \sin x = d = r \cdot \cos(x+\Delta x) \geq \Delta x \cdot \cos(x+\Delta x).$$

The primitive of cosine can alternatively be found by translocation rule.

$$I_{\cos}(x) = I_{\sin}(x+\pi/\ \ ) = -\cos(x+\pi/\ \ ) = -(-\sin x) = \sin x.$$

Altogether follow from linearity, that $I_{\sin} = -\cos$, $I_{\cos} = \sin$, $I_{-\sin} = \cos$, $I_{-\cos} = -\sin$. With substitution rule (to be proved in the next section) follows

$$I_{\sin(ax+b)}(x) = \frac{-}{a} \cdot \cos(a \cdot x + b).$$

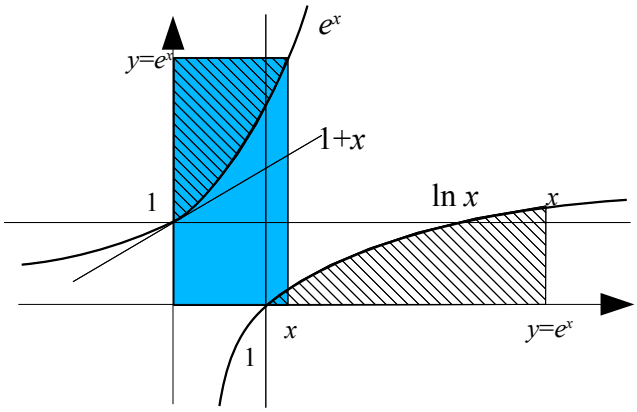
7.2.3 Derivatives of sine and cosine

For the derivatives as inversion of integration follows

$$D_{\sin}(x) = \cos x\,;\, D_{\cos}(x) = -\sin x.$$

7.3 Exponential function and natural logarithm

Two other important function are exponential function with base e (*Euler* number), also called e-function, and its inverse function the natural logarithm, written as ln x. Instead of e^x we write also exp x, so that primitive and derivative can be written more easily.



(19)

7.3.1 Primitive of *e*-function

The primitive of the e-function is the e-function itself. This follows from two properties of exponential functions in general and the e-function specially.

1. $e^{a+b} = e^a \cdot e^b$
2. $e^x \geq 1+x$

The second point means that e-function is convex from the bottom and the ascent for $x = 0$ is 1. We can also say, that the e-function is that function that is exponential and has the corresponding ascent in origin, and the base e is to be determined in another way that we will show later in chapter 10.1. The primitive of e^x is the same function $I_{\exp}(x) = \exp(x) = e^x$.

$$I_{e^x}(x) = I_{\exp}(x) = e^x .$$

To prove this, we have to prove the inequalities

$$e^x \cdot \Delta x \leq e^{x+\Delta x} - e^x = e^x \cdot \left(e^{\Delta x} - \quad \right) \leq e^{x+\Delta x} \cdot \Delta x . \tag{20}$$

Since $z+1 \leq e^z$ for each z, we substitute z by Δx and get $\Delta x+1 \leq e^{\Delta x}$, thus $\Delta x \leq e^{\Delta x}-1$, and $e^x \cdot \Delta x \leq e^x \cdot e^{\Delta x} - e^x = e^{x+\Delta x} - e^x$. This proves the first inequality.

In $z+1 \leq e^z$, we replace z with $-\Delta x$: $1-\Delta x \leq e^{-\Delta x}$, $1-e^{-\Delta x} \leq \Delta x$, $e^{\Delta x} - e^{\Delta x} \cdot e^{-\Delta x} = e^{\Delta x} - 1 \leq e^{\Delta x} \cdot \Delta x$, and $e^x \cdot e^{\Delta x} - e^x = e^{x+\Delta x} - e^x \leq e^x e^{\Delta x} \cdot \Delta x = e^{x+\Delta x} \cdot \Delta x$; this proves the second inequality.

7.3.2 Derivative of e-function

Analogous the derivative of e^x is $D_{\exp}(x) = e^x$.

$$D_{e^x}(x) = D_{\exp}(x) = e^x .$$

7.3.3 Primitive of natural logarithm

The inverse function of *e*-Function is the natural logarithm $\ln x$; we have $e^{\ln x} = x = \ln e^x$. The primitive of the natural logarithm can simply be derived from geometric considerations and the primitive of e^x. In graphic (19) the area beneath the curve e^x is just $e^x - e^0 = e^x - 1$, so the striped area is $x \cdot e^x - (e^x - 1) = e^x \cdot (x-1)+1$. Also, the area beneath the curve $\ln x$ in interval $[1,y]$ has the same size, with $x = \ln y$: $I_{\ln}(y) = y \cdot (\ln y - 1)+1$. Since primitives do not depend on constant factors, we have (replacing y by x)

$$I_{\ln}(x) = x \cdot (\ln x - \quad).$$

This method can be used for all functions, the primitive of the inverse function of which are known. We give more details in (35).

For completeness we prove this also with inequalities (4) directly. To do so, we have to prove

$$\Delta x\cdot\ln x\leq(x+\Delta x)\cdot(\ln(x+\Delta x)-\ )-x\cdot(\ln x-\ )\leq\Delta x\cdot\ln(x+\Delta x). \tag{21}$$

From graphic (19) follows the plot of the logarithm. As inverse function to *e*-Function, the ascent of the logarithm in $x = 1$ is 1, so the tangent is $y = x–1$. For any positive z follows ln z ≤ z–1.

Replacing $z = x/(x+\Delta x)$, follows

$$\ln\frac{x}{x+\Delta x}=\ln x-\ln(x+\Delta x)\leq\frac{x}{x+\Delta x}-\ =\frac{-\Delta x}{x+\Delta x},$$

reordering and multiplication with positive numerator yields

$$\leq(x+\Delta x)\cdot\ln(x+\Delta x)-(x+\Delta x)\cdot\ln x-\Delta x=(x+\Delta x)\cdot\ln(x+\Delta x)-x\cdot\ln x-\Delta x\cdot\ln x-\Delta x.$$

From this follows the left inequality of (21) by reordering and adding $x–x$:

$$\Delta x\cdot\ln x\leq(x+\Delta x)\cdot\ln(x+\Delta x)-x-\Delta x-x\cdot\ln x+x=(x+\Delta x)\cdot(\ln(x+\Delta x)-\ )-x\cdot(\ln x-\ ).$$

Since $z–1 \geq \ln z$ for each positive z, follows with $z = 1+\Delta x/x$

$$\frac{\Delta x}{x}\geq\ln(\ +\frac{\Delta x}{x})=\ln(x+\Delta x)-\ln x.$$

Multiplication with positive numerator x and adding some terms yields

$$\Delta x\cdot\ln(x+\Delta x)\geq x\cdot\ln(x+\Delta x)+\Delta x\cdot\ln(x+\Delta x)-x-\Delta x-x\cdot\ln x+x.$$

Then follows the other inequality (21).

7.3.4 Derivative of the natural logarithm

Derivative of natural logarithm is $D_{\ln}(x) = 1/x$; this follows from geometric considerations as well. At first we give an algebraic proof and show with inequalities (5) that $I_{1/x} = \ln x$.

Since $1/x$ is monotonously ascending we have to show that

$$\frac{\Delta x}{x}\geq\ln(x+\Delta x)-\ln x\geq\frac{\Delta x}{x+\Delta x}.$$

From (21) follows

$$(x+\Delta x)\cdot(\ln(x+\Delta x)-\ )-x\cdot(\ln x-\ )=$$
$$x\cdot\ln(x+\Delta x)+\Delta x\cdot\ln(x+\Delta x)-\Delta x-x\cdot\ln x\leq\Delta x\cdot\ln(x+\Delta x)$$

or

$$x\cdot\ln(x+\Delta x)-x\cdot\ln x\leq\Delta x$$

from which follows the left hand side of the inequality. Also from (21) follows

$$\Delta x\cdot\ln x\leq(x+\Delta x)\cdot(\ln(x+\Delta x)-\ )-x\cdot(\ln x-\ )=(x+\Delta x)\cdot\ln(x+\Delta x)-\Delta x-x\cdot\ln x$$

or

$$\Delta x\leq(x+\Delta x)\cdot\ln(x+\Delta x)-x\cdot\ln x-\Delta x\cdot\ln x=(x+\Delta x)\cdot\ln(x+\Delta x)-(x+\Delta x)\cdot\ln x,$$

this proves the other inequality.

A geometric proof follows from graphic (19). There the tangent to e^x in point x with ascent $D_{exp}(x)$ is plotted, and because of the properties of the inverse function follows that the derivative of the natural logarithm in point $y = e^x$ is just the reciprocal value of the derivative of e^x in point x.

Thus we get simply

$$D_{\ln}(y)=\frac{}{D_{e^x}(x)}=\frac{}{e^x}=\frac{}{y}$$

this proves the statement as well.

This technique can be generalized to all functions, the inverse of which are known. It will be presented more clearly with the inversion rule (33).

8 Rules for integration and differentiation

8.1 Product rule

There are some rules for differentiation and integration that can be used to derive from known derivatives or primitives other ones. Because of linearity follows for the sum of two derivatives

$D_{f+g} = D_f + D_g$;

or the derivative of a multiple of a derivative

$D_{a\cdot f} = a\cdot D_f$.

Similar rules hold for product and quotient of two function; this will now be derived.

8.1.1 Product rule for differentiation

If the derivative of two functions f and g is known, then the derivative of the product of both functions can be derived. Let us assume both functions f and g are monotonously ascending

$f(x) \leq f(x+\Delta x)$,

$g(x) \leq g(x+\Delta x)$,

then follows

$$(f(x+\Delta x)-f(x))\cdot(g(x+\Delta x)-g(x))\geq \quad ,$$

that means

$$f(x+\Delta x)\cdot g(x+\Delta x)-f(x+\Delta x)\cdot g(x)-f(x)\cdot g(x+\Delta x)+f(x)\cdot g(x)\geq \quad , \tag{22}$$

or

$$f(x)\cdot(g(x+\Delta x)-g(x))+g(x)\cdot(f(x+\Delta x)-f(x))\leq f(x+\Delta x)\cdot g(x+\Delta x)-f(x)\cdot g(x).$$

We have defined

$$\begin{aligned} D_f(x)\cdot\Delta x &\leq f(x+\Delta x)-f(x)\leq D_f(x+\Delta x)\cdot\Delta x, \\ D_g(x)\cdot\Delta x &\leq g(x+\Delta x)-g(x)\leq D_g(x+\Delta x)\cdot\Delta x. \end{aligned} \tag{23}$$

Then follows

$$\begin{aligned} f(x)\cdot D_g(x)\cdot\Delta x+g(x)\cdot D_f(x)\cdot\Delta x &= (f(x)\cdot D_g(x)+g(x)\cdot D_f(x))\cdot\Delta x\leq \\ &\leq f(x)\cdot(g(x+\Delta x)-g(x))+g(x)\cdot(f(x+\Delta x)-f(x))\leq \\ &\leq f(x+\Delta x)\cdot g(x+\Delta x)-f(x)\cdot g(x). \end{aligned}$$

Also follows from (22)

$$\begin{aligned} f(x+\Delta x)\cdot g(x+\Delta x)-f(x)\cdot g(x)&\leq \\ &\leq f(x+\Delta x)\cdot(g(x+\Delta x)-g(x))+g(x+\Delta x)\cdot(f(x+\Delta x)-f(x))\leq \\ &\leq f(x+\Delta x)\cdot D_g(x+\Delta x)\cdot\Delta x+g(x+\Delta x)\cdot D_f(x+\Delta x)\cdot\Delta x. \end{aligned}$$

Altogether we get

$$\begin{aligned} (f(x)\cdot D_g(x)+g(x)\cdot D_f(x))\cdot\Delta x&\leq \\ &\leq f(x+\Delta x)\cdot g(x+\Delta x)-f(x)\cdot g(x)\leq \\ &\leq (f(x+\Delta x)\cdot D_g(x+\Delta x)+g(x+\Delta x)\cdot D_f(x+\Delta x))\cdot\Delta x. \end{aligned}$$

Thus follows the so called *Product rule.*

$$D_{f\cdot g}(x)=f(x)\cdot D_g(x)+D_f(x)\cdot g(x). \quad \textit{Product rule} \tag{24}$$

As an example let us find the derivative of the product of sine and cosine.

$$D_{\sin x\cdot\cos x} = \sin x \cdot D_{\cos x} + D_{\sin x} \cdot \cos x = \sin x \cdot - \sin x + \cos x \cdot \cos x = \cos^2 x - \sin^2 x = 1-2\cdot\sin^2 x.$$

8.1.2 Partial Integration

The product rule can be used to integrate as well, by reordering we get

$$f(x)\cdot D_g(x)=D_{f\cdot g}(x)-g(x)\cdot D_f(x).$$

If the primitive of $g(x)\cdot D_f(x)$ is known, then $f(x)\cdot D_g(x)$ can be integrated as well.

$$I_{f(x)\cdot D_g(x)}=f(x)\cdot g(x)-I_{g(x)\cdot D_f(x)}. \quad \textit{Partial Integration} \tag{25}$$

For example let be $f(x) = x$, $g(x) = -\cos x$, then $D_f(x) = 1$, $D_g(x) = \sin x$, and $I_{\cos}(x) = \sin x$; then follows

$$I_{x\cdot\sin x}=-x\cdot\cos x-I_{-\cos}(x)=-x\cdot\cos x+\sin x.$$

This can be checked by finding the derivative of the primitive again.

$$D_{-x\cdot\cos x+\sin x}=D_{-x\cdot\cos x}+D_{\sin x}=- \quad \cdot\cos x+x\cdot\sin x+\cos x=x\cdot\sin x.$$

8.2 Reciprocal rule

The reciprocal rule determines the derivative of the reciprocal $1/f$ of a function f. Let f be positive and never zero: $f(x) > 0$. Let $1/f$ be monotonously ascending. We have the relationship

$$(f(x+\Delta x)-f(x))=f(x+\Delta x)\cdot f(x+\Delta x)- \quad \cdot f(x+\Delta x)\cdot f(x)+f(x)\cdot f(x)\geq \quad ,$$
$$f(x+\Delta x)\cdot(f(x+\Delta x)-f(x))\geq f(x+\Delta x)\cdot f(x)-f(x)\cdot f(x),$$
$$-\frac{f(x+\Delta x)-f(x)}{f\ (x)}\leq\frac{}{f(x+\Delta x)}-\frac{}{f(x)}.$$

Analogous follows

$$(f(x+\Delta x)-f(x))=f(x+\Delta x)\cdot f(x+\Delta x)- \quad \cdot f(x+\Delta x)\cdot f(x)+f(x)\cdot f(x)\geq \quad ,$$
$$f(x+\Delta x)\cdot f(x+\Delta x)-f(x+\Delta x)\cdot f(x)\geq f(x+\Delta x)\cdot f(x)-f(x)\cdot f(x),$$
$$\frac{}{f(x+\Delta x)}-\frac{}{f(x)}\geq-\frac{f(x+\Delta x)-f(x)}{f\ (x+\Delta x)}.$$

Altogether we get

$$-\frac{f(x+\Delta x)-f(x)}{f\ (x)}\leq\frac{}{f(x+\Delta x)}-\frac{}{f(x)}\leq-\frac{f(x+\Delta x)-f(x)}{f\ (x+\Delta x)}.$$

The reciprocal rule follows

$$D_{1/f}(x)=\frac{-D_f(x)}{f\ (x)}. \quad \textit{Reciprocal rule} \tag{26}$$

As an example for the reciprocal rule (27) let us compute the derivative with negative exponent. Let be $f(x) = x^k$, where $k > 0$. then

$$D_{x^{-k}}(x)=D_{1/f}(x)=\frac{-D_f(x)}{f\ (x)}=\frac{-D_{x^k}(x)}{x^{\mathrm{k}}}=\frac{-k\cdot x^{k-}}{x^{\mathrm{k}}}=-k\cdot x^{-k-}.$$

Die power rule (17) holds for negative integer exponents as well. (27)

8.3 Quotient rule

The product rule follows from the product of a function f and the reciprocal of a function g.

$$D_{f/g}(x)=D_f(x)/g(x)+f(x)\cdot D_g(x)=\frac{D_f(x)}{g(x)}-\frac{f(x)\cdot D_g(x)}{g\ (x)}=\frac{D_f(x)\cdot g(x)-f(x)\cdot D_g(x)}{g\ (x)}.$$

The quotient rule is therefore

$$D_{f/g}(x)=\frac{D_f(x)\cdot g(x)-f(x)\cdot D_g(x)}{g\ (x)}.\quad \textit{Quotient rule} \tag{28}$$

As an example for the quotient rule (28) we derive the quotient of sine and cosine, that is the tangent.

$$D_{\sin/\cos}(x)=\frac{D_{\sin}(x)\cdot\cos x-\sin x\cdot D_{\cos}(x)}{\cos\ x}=\frac{\cos x\cdot\cos x-\sin x\cdot -\sin x}{\cos\ x}=\frac{\quad}{\cos\ x}.$$

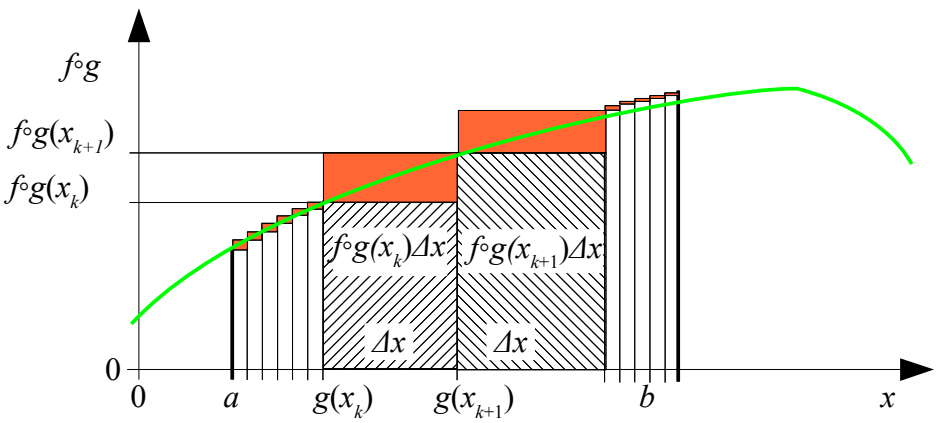
As another example let us compute the derivative of $\sin(x)/x$

$$D_{\sin x/x}(x)=\frac{x\cdot D_{\sin}(x)-D_x(x)\cdot\sin x}{x}=\frac{x\cdot\cos x-\sin x}{x}=\frac{\cos x}{x}-\frac{\sin x}{x}.$$

8.4 Substitution rule

The substitution rule analyses functions like $f(g(x))$; this can be written as $f\circ g(x)$, where $f\circ g$ can be looked at as a new function h: $f(g(x)) = f\circ g(x) = h(x)$. For functions the presentations $f(g(x))$ and $f\circ g(x)$ are equivalent. However, if $f(g(x))$ will be integrated, intregration is be done for the variable x, so that the formula for I_f cannot be used as a primitive for $f(g(x))$. The formula $I_f(g(x))$ means the function I_f with parameter $g(x)$, while $I_{f\circ g}(x)$ means the primitive of $f\circ g$, what can be completely different from $I_f(g(x))$.

From the diagram



follows

$$\begin{aligned} f(g(x))\cdot\big(g(x+\Delta x)-g(x)\big)\le I_f(g(x+\Delta x))-I_f(g(x))\le \\ \le f(g(x+\Delta x))\cdot\big(g(x+\Delta x)-g(x)\big). \end{aligned} \tag{29}$$

Using the definition of a derivative (23) yields

$$\begin{aligned} f(g(x))\cdot D_g(x)\cdot\Delta x &\le f(g(x))\cdot\big(g(x+\Delta x)-g(x)\big)\cdot\Delta x\le \\ &\le I_f(g(x+\Delta x))-I_f(g(x))\le \\ &\le f(g(x+\Delta x))\cdot\big(g(x+\Delta x)-g(x)\big)\cdot\Delta x\le \\ &\le f(g(x+\Delta x))\cdot D_g(x+\Delta x)\cdot\Delta x. \end{aligned} \tag{30}$$

This rule states that the primitive of the function $f\circ g\cdot D_g$ is the primitive of f with parameter $g(x)$. Thus from (30) follows the substitution rule for integration

$$I_{f\circ g\cdot D_g}(x)=I_f(g(x)). \quad \textit{Substitution rule for Integration} \tag{31}$$

For example let be $g(x) = a\cdot x+b$, then $D_g(x) = a$, thus $I_f(a\cdot x+b)$ is the integral of $f\circ g(x)\cdot a$, or because of linearity of integration follows by reducing

$$I_{f(a\cdot x+b)}(x)=I_{f\circ g}(x)=\frac{}{a}\cdot I_f(a\cdot x+b).$$

Substitution rule for integration is possible only for very special functions. More general rules follow for differentiation.

8.4.1 Substitution rule for differentiation

For the derivative we assume that the functions D_f and D_g are given, where

$$D_f(x)\cdot\Delta x\le f(x+\Delta x)-f(x)\le D_f(x+\Delta x)\cdot\Delta x,$$
$$D_g(x)\cdot\Delta x\le g(x+\Delta x)-g(x)\le D_g(x+\Delta x)\cdot\Delta x.$$

From (29) follows after renaming(f becomes D_f, I_f becomes f)

$$\begin{aligned} D_f(g(x))\cdot D_g(x)\cdot\Delta x &\le D_f(g(x))\cdot\big(g(x+\Delta x)-g(x)\big)\le \\ &\le f(g(x+\Delta x))-f(g(x))\le \\ &\le D_f(g(x+\Delta x))\cdot\big(g(x+\Delta x)-g(x)\big)\le D_f(g(x+\Delta x))\cdot D_g(x+\Delta x)\cdot\Delta x. \end{aligned}$$

Thus follows the substitution rule differentiation

$$D_{f\circ g}(x)=D_f(g(x))\cdot D_g(x). \quad \textit{Substitution rule for differentiation} \tag{32}$$

Technically, $g(x)$ is replaced by a variable $z = g(x)$ and then the derivative of f to z is computed. This is to be multiplied by the derivative of z to x to get the derivative of $f(g(x))$. Pseudoformally the is justified by the differential quotient

$$D_{f\circ g}(x)=D_f(z)\cdot D_z(x)=\frac{df}{dz}\cdot\frac{dz}{dx}=\frac{df}{dx}.$$

In this formula the differential dz reduce. However, this is no consistent definition.

For example for any function f and $a(x) = a\cdot x+b$ follows as derivative of $f(a(x)) = f(a\cdot x+b)$

$$D_{f\circ a}(x)=D_f(a(x))\cdot D_a(x)=D_f(a\cdot x+b)\cdot a=a\cdot D_f(a\cdot x+b).$$

8.5 Inversion rule

Let be g the inverse function of f, $f(g(x)) = x$ for each x; sometimes the notation f^{1} is used for the inverse function f. The inverse function is defined uniquely only, when f does never attain the same value twice in the interval considered, thus $f(x) \ne f(y)$ if $x \ne y$. Thus only strictly monotonous functions are considered here.

8.5.1 Inversion rule for differentiation

The inversion rule for derivatives follows simply from the substitution rule. From substitution rule (32) follows with $D_{f\circ g}(x) = D_x(\mathrm{x}) = 1$, that $D_{f\circ g}(x) = D_f(g(x))\cdot D_g(\mathrm{x}) = 1$, so that we have

$$D_g(x)=\frac{}{D_f(g(x))}. \quad \text{Inversion ruel} \tag{33}$$

For example let be $f(x) = e^x$, then $f^{-1}(x) = \ln x$, thus

$$D_{\ln}(x)=\frac{}{D_f(\ln x)}=\frac{}{e^{\ln x}}=\frac{}{x}.$$

Another application of inverse functions is that for $f(x) = x^k$, which is obviously $g(x) = x^{1/k}$.

$f(g(x)) = (g(x))^k = (x^{1/k})^k = x^{k/k} = x^1 = x.$

Then we get for the derivative

$$D_{x^{/k}}(x)=\frac{}{D_{x^k}(x^{/k})}=\frac{}{k\cdot(x^{/k})^{k-}}=\frac{}{k}\cdot x^{(\ -k)/k}=\frac{}{k}\cdot x^{\ /k-}.$$

Thus the power rule (17) follows also for rational exponents. As can be shown by substitution rule this holds for all rational exponents. For arbitrary integer numbers m and k ($k\neq 0$) follows

$$D_{x^{m/k}}(x)=D_{(x^{/k})^m}(x)=D_{(x)^m}(x^{/k})\cdot D_{x^{/k}}(x)=m\cdot(x^{/k})^{m-}\cdot\frac{}{k}\cdot x^{\ /k-}=\frac{m}{k}\cdot x^{(m-\ )/k+\ /k-}=\frac{m}{k}\cdot x^{m/k-}.$$

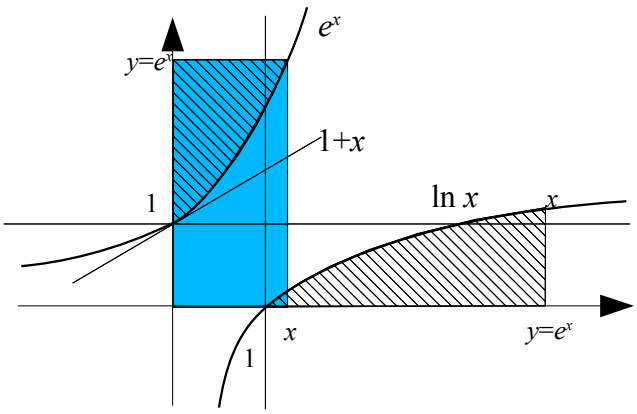
Thus the power rule (17) is valid for all (rational) exponents.

$$D_{x^q}(x)=q\cdot x^{q-},$$
$$I_{x^q}(x)=\frac{}{q+}\cdot x^{q+}, \quad \textit{Power rule} \tag{34}$$

for each rationale q, with exception $q = -1$, the primitive of which is $\ln x$.

8.5.2 Inversion rule for integration

Another rule follows from simple geometric considerations.



The diagram shows the principal plot of the e-function and the natural logarithm, where the striped areas have the same size. In general follows for an arbitrary monotonously ascending function f, with value $b = f(0)$ at origin, that the area in the interval $[b,y]$ is, where $y = f(x)$, $g(y) = x$,

$$I_g(y)-I_g(b)=x\cdot f(x)-(I_f(x)-I_f(\ \ ))=g(y)\cdot y-(I_f(g(y))-I_f(\ \ )),$$

or after renaming of y to x and dropping constant terms we get as primitive function

$$I_g(x)=g(x)\cdot x-I_f(g(x)).\quad \text{Inversion rule for primitive} \tag{35}$$

To show this algebraically, we use rules for addition and substitution (31)

$$I_g(x)+I_f(g(x))=I\left[\,g(x)+f(g(x))\cdot D_g(x)\right]=I\left[\,g(x)+x\cdot D_g(x)\right].$$

Here $I[f](x)$ stands for $I_f(x)$, to keep the formulae clear. Since we assume $f(g(x)) = x$ follows from product rule $D_{g\cdot h}(x) = D_g(x)\cdot h(x)+D_h(x)\cdot g(x)$, with $h(x) = x$ and $D_x(x) = 1$

$$g(x)\cdot x=I[D_{g\cdot x}](x)=I\left[\,g(x)+x\cdot D_g(x)\right](x)=I_g(x)+I_f(g(x)).$$

This proves the inversion rule for primitives (35) algebraically. As proved already, follows for the natural logarithm

$$I_{\ln}(x)=\ln(x)\cdot x-I_{e^x}(\ln(x))=\ln(x)\cdot x-e^{\ln(x)}=\ln(x)\cdot x-x.$$

Other applications will be given below, for the inverse functions of trigonometric function.

8.5.3 Quotient rule for primitives

In very special cases also integration can be done by a quotient rule. The primitive of a function that ca be displayed as $D_g(x)/g(x)$ is ln(g(x)). We use substitution rule (31)

$$I_{f\circ g\cdot D_g}(x)=I_f(g(x)).$$

If $f(x) = 1/x$, with primitive $I_f = \ln x$, then the primitive of

$$\frac{D_g(x)}{g(x)}=D_g(x)\cdot\frac{}{g(x)}=D_g(x)\cdot f\circ g(x)$$

is the function

$$I_{D_g/g}(x)=I_{D_g\cdot f\circ g}(x)=I_f(g(x))=I_{\ /x}(g(x))=-\ln(g(x)). \tag{36}$$

8.6 Tangent and Cotangent

Tangent and cotangent are defined as

$\tan x = \sin x / \cos x$,

$\cot x = \cos x / \sin x$,

and have a close relationship

$\tan x = 1 / \cot x$.

8.6.1 Integral

The primitive of $\tan x$ is $-\ln(\cos x)$. We consider only the intervall $[0,\pi/2)$, since tangent has a pole for $x = \pi/2$. Because of $\tan -x = -\tan x$ the results hold in the interval $(-\pi/2\ ,\pi/2)$ as well. We use substitution rule (31)

$$I_{f\circ g\cdot D_g}(x)=I_f(g(x)).$$

Is $f(x) = -1/x$, with the integral $I_f = -\ln x$, and g(x) = $\cos x$ with $D_{\cos} = -\sin$, then the primitive of

$$\tan x=\frac{\sin x}{\cos x}=-D_{\cos x}\cdot\frac{}{\cos x}=D_g(x)\cdot f\circ g(x)$$

is the function

$$I_{\tan}(x)=I_{D_g\cdot f\circ g}(x)=I_f(g(x))=I_{-\ /x}(\cos x)=-\ln(\cos x).$$

Because of cot x = cos x/sin x and $D_{\sin}$ = cos follows in the same way

$$I_{\cot}(x)=\ln(\sin x).$$

8.6.2 Derivative

The derivative of tangent follows from quotient rule for differentiation.

$$D_{\tan}(x)=D_{\sin/\cos}(x)=\frac{\cos x\cdot\cos x-\sin x\cdot-\sin x}{\cos\ x}=\frac{}{\cos\ x}.$$

Analogous follows for the derivative of the cotangent from the quotient rule for differentiation.

$$D_{\cot}(x)=D_{\cos/\sin}(x)=\frac{-\sin x\cdot\sin x-\cos x\cdot\cos x}{\sin\ x}=\frac{-}{\sin\ x}.$$

This can also be shown by reciprocal rule (26).

$$D_{\cot}(x)=D_{\ /\tan}(x)=\frac{-D_{\tan}(x)}{\tan\ x}=\frac{-\ \ /\cos\ x}{\sin\ x/\cos\ x}=\frac{-}{\sin\ x}.$$

8.6.3 Direct Proves

Although many formulae can be proved by rules derived in this section, they can usually proved directly. For example for tangent we have to prove because of our standard definition (4,5)

$$\frac{}{\cos\ x}\cdot\Delta x\leq\tan(x+\Delta x)-\tan x\leq\frac{}{\cos\ (x+\Delta x)}\cdot\Delta x.$$

Multiplying the numerators yields

$$\begin{aligned}\cos\ (x+\Delta x)\cdot\Delta x&\leq\sin(x+\Delta x)\cdot\cos(x+\Delta x)\cdot\cos\ x-\sin x\cdot\cos x\cdot\cos\ (x+\Delta x)=\\&=\cos x\cdot\cos(x+\Delta x)\cdot(\sin(x+\Delta x)\cdot\cos x-\sin x\cdot\cos(x+\Delta x))=\\&=\cos x\cdot\cos(x+\Delta x)\cdot\sin(\Delta x)\leq\cos\ x\cdot\Delta x.\end{aligned}$$

We used the trigonometric theorem for addition sin ((x+Δx)–x). Dividing the right inequality by the positive term cos x yields

$$\sin\Delta x\cdot\cos(x+\Delta x)\leq\cos x\cdot\Delta x;$$

this is true since cos(x+Δx) < cos x, and sin Δx < Δx. For the left hand side inequality we have

$$\cos(x+\Delta x)\cdot\Delta x\leq\sin\Delta x\cdot\cos x.$$

To prove this we use the inequalities for cosine (18)

$$\cos(x+\Delta x)\cdot\Delta x\leq\sin(x+\Delta x)-\sin x=\sin x\cdot(\cos\Delta x-\ \ )+\cos x\cdot\sin\Delta x\leq\cos x\cdot\sin\Delta x,$$

since for positive Δx always holds cos Δx < 1.

This shows that any direct prove can become quite more complicated, however, should always be possible, i.e. any formula should be provable by our standard definition (4,5), as well.

8.6.4 Antitrigonometric functions

We call inverse function of a trigonometric function like sine or cosine the arcus functions: arcsin, arccos, arctan arccot. From formula (33)

$$D_g(x)=\frac{}{D_f(g(x))}$$

follows for arcsin

$$D_{\arcsin}(x)=\frac{}{D_{\sin}(\arcsin(x))}=\frac{}{\cos(\arcsin(x))}=\frac{}{\sqrt{-\sin\ (\arcsin(x))}}=\frac{}{\sqrt{-x}},$$

and for arctan

$$D_{\arctan}(x)=\frac{}{D_{\tan}(\arctan(x))}=\cos\ (\arctan(x))=\frac{}{\sqrt{+\tan\ (\arctan(x))}}=\frac{}{\sqrt{+x}}.$$

Also primitives follow from inversion rule (35), that read

$$I_g(x)=g(x)\cdot x-I_f(g(x)).$$

For the primitive of arcsin we get

$$I_{\arcsin}(x)=\arcsin(x)\cdot x-I_{\sin}(\arcsin(x))=\arcsin(x)\cdot x+\cos(\arcsin(x))=\arcsin(x)\cdot x+\sqrt{-x}.$$

For the primitive of arccos we get

$$I_{\arccos}(x)=\arccos(x)\cdot x-I_{\cos}(\arcsin(x))=\arccos(x)\cdot x-\sin(\arccos(x))=\arccos(x)\cdot x-\sqrt{-x}.$$

For primitive of arctan we get

$$I_{\arctan}(x)=\arctan(x)\cdot x-I_{\tan}(\arctan(x))=\arctan(x)\cdot x+\ln(\cos(\arctan(x))).$$

The last formula can be simplified by some trigonometric relationships. We have

$$\tan x=\frac{\sin x}{\cos x}=\frac{\sqrt{-\cos\ x}}{\cos x},$$

$$\tan\ x\cdot\cos\ x=\ -\cos\ x,$$

$$\cos x=\frac{}{\sqrt{+\tan\ x}}.$$

and then follows

$$\cos\arctan x=\frac{}{\sqrt{+x}},$$

if arctan is used as parameter. Since logarithm of a root is the logarithm of the half value, and reciprocal value is the negative logarithm we get as primitive of acrtan

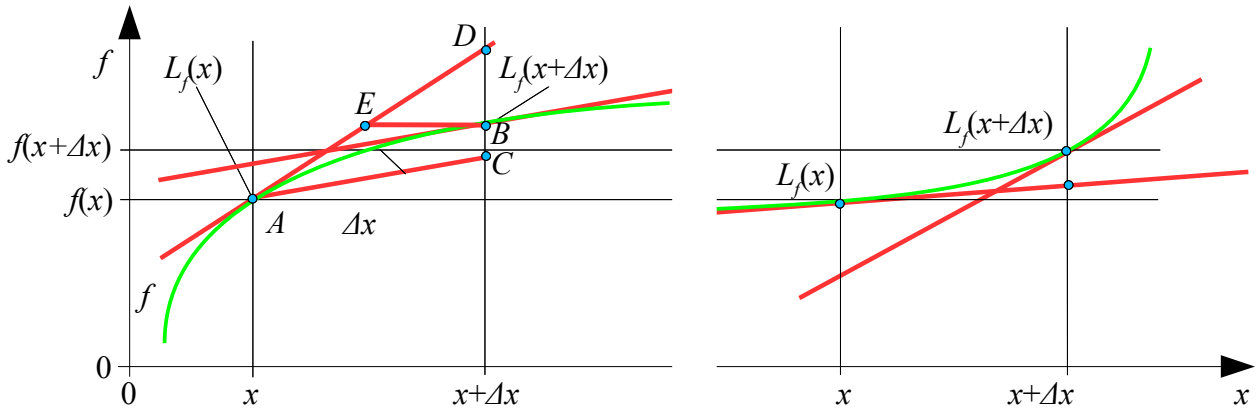
$$I_{\arctan}(x)=\arctan(x)\cdot x--\cdot\ln(\ +x\ ).$$

9 Length of a curve

To find the length of a curve the standard mathematics derives the formala

$$L_f(z)=\int^{z}\sqrt{\quad+D_f(x)}\;dx$$

We will derive this formala by the inequalities (4, 5). The following diagram for a monotonously ascending curve that is convex from top shows that the length AC is shorter then the (green) curve of the function f from A to B, since the length AC is shorter than the secant from A to B. The length AC can be computed from $\Delta x/\cos\alpha$, where α is the angle of the tangent in B at the curve of the function f in point $x+\Delta x$. Analogous the length AD is longer than the curve of the function f from A to B, since the length $AE+EB$ is always longer than then this curve and itself is longer than AD. The length AD can be computed from $\Delta x/\cos\beta$, where β is the angle of the tangent in A at the curve of the function f in point x.



If the curve is convex from the bottom follows analogously that $\Delta x/\cos\beta$ is shorter than the curve between two points, when β is the angle of the tangent to the curve of the function f in point x, or that $\Delta x/\cos\alpha$ is longer than the curve between two points if α is the angle of the tangent at the curve of the function f in point $x+\Delta x$. Again the length of the curve can be limited between two straight lines, und again the total length can be found by the sum of the partial lengths. To determine the deviation, the differences of the straight lines are to be summed, thus

$$\sum_{i=}^{N-}\Delta x/\cos\alpha_{i+}-\Delta x/\cos\alpha_i=\Delta x/\cos\alpha_N-\Delta x/\cos\alpha=\Delta x\cdot\left(\frac{}{\cos\alpha_N}-\frac{}{\cos\alpha}\right).$$

The deviation is proportional to the step width Δx, where we only have to assume that the angle of the ascent is less than 90°.

The length $\Delta x/\cos\alpha$ can be found from the tangent of the angle α of the tangent to the horizontal line, where we use the identity

$$\cos x=\frac{}{\sqrt{\quad+\tan\;x}}$$

and the length of the tangent in the section $[x, x+\Delta x]$ is

$$L_t=\frac{\Delta x}{\cos x}=\Delta x\cdot\sqrt{\quad+\tan\;x}=\Delta x\cdot\sqrt{\quad+D_f(x)}.$$

For a curve that is convex from the bottom follow the inequalities

$$\Delta x\cdot\sqrt{\quad+D_f(x)}\le L_f(x+\Delta x)-L_f(x)\le\Delta x\cdot\sqrt{\quad+D_f(x+\Delta x)}. \tag{37}$$

For a curve that is convex from the top follow the inequalities

$$\Delta x\cdot\sqrt{\quad+D_f(x+\Delta x)}\leq L_f(x+\Delta x)-L_f(x)\leq\Delta x\cdot\sqrt{\quad+D_f(x)}. \tag{38}$$

We can assume that L_f is the primitive to $\sqrt{\quad+D_f(x)}$ and come to the same result as in standard mathematics. If there is a deviation between the real length of the curve and the function L_f follows from reducing the step with Δx that the function L_f does not obey (37) and (38) from which uniqueness of length function L_f follows.

9.1.1 Example

The function $f(x) = a\cdot x$ has the derivative $D_f(x) = a$. Then its length in interval $[0,x]$ is given by the function

$$L_f(x)=\sqrt{\quad+a}\cdot x.$$

Then follows with the first (and analogous the second) inequalities

$$\begin{aligned}&\Delta x\cdot\sqrt{\quad+D_f(x)}=\Delta x\cdot\sqrt{\quad+a}\leq\\&\quad\leq L_f(x+\Delta x)-L_f(x)=\sqrt{\quad+a}\cdot(x+\Delta x)-\sqrt{\quad+a}\cdot x=\sqrt{\quad+a}\cdot\Delta x\leq\\&\quad\leq\Delta x\cdot\sqrt{\quad+D_f(x+\Delta x)}=\Delta x\cdot\sqrt{\quad+a}.\end{aligned}$$

9.1.2 Example

The function

$$f(x)=-\cdot x^{\ /}$$

hat the derivative $D_f(x) = x^{1/2}$. Then its length in the interval $[0,x]$ is given by the function

$$L_f(x)=-\cdot(\quad+x)^{\ /}.$$

Since this function is monotonously ascending and convex from the bottom inequalities (37) must hold.

$$\begin{aligned}&\Delta x\cdot\sqrt{\quad+D_f(x)}=\Delta x\cdot\sqrt{\quad+x}\leq\\&\quad\leq L_f(x+\Delta x)-L_f(x)=-\cdot(\quad+x+\Delta x)^{\ /}--\cdot(\quad+x)^{\ /}\leq\\&\quad\leq\Delta x\cdot\sqrt{\quad+D_f(x+\Delta x)}=\Delta x\cdot\sqrt{\quad+x+\Delta x}.\end{aligned}$$

We substitute $z = 1+x$ and get for the left inequality

$$\Delta x\cdot\sqrt{z}=\Delta x\cdot z' \le -\cdot(z+\Delta x)' - -\cdot z',$$

$$\cdot\Delta x \le \cdot\sqrt{\frac{(z+\Delta x)}{z}} - \cdot z,$$

$$\cdot\Delta x + \cdot z \le \cdot\sqrt{\frac{(z+\Delta x)}{z}},$$

$$\cdot\Delta x + \cdot z + \cdot z\cdot\Delta x \le \cdot\frac{(z+\Delta x)}{z} = \cdot(z + \cdot z\cdot\Delta x + \cdot\Delta x + \frac{\Delta x}{z}),$$

$$\le \cdot\Delta x + \cdot\frac{\Delta x}{z} = \cdot\Delta x + \cdot\frac{\Delta x}{+x}.$$

for positive x and Δx this inequality holds. For the right inequality we must prove that

$$-\cdot(+x+\Delta x)' - -\cdot(+x)' = -\cdot(z+\Delta x)' - -\cdot z' \le \Delta x\cdot\sqrt{z+\Delta x},$$

$$\cdot(z+\Delta x)' - \cdot z' \le \cdot\Delta x\cdot\sqrt{z+\Delta x},$$

$$\cdot(z+\Delta x) - \cdot\sqrt{\frac{z}{z+\Delta x}} \le \cdot\Delta x,$$

$$\cdot(z+\Delta x) - \cdot\Delta x = \cdot z - \Delta x \le \cdot\sqrt{\frac{z}{z+\Delta x}}.$$

If $\Delta x > 2\cdot z = 2+2\cdot x$, then the left hand side is negative, thus the inequality is correct. Thus if $\Delta x \le 2\cdot z$ then both sides are positive and we can square.

$$\cdot z + \Delta x - \cdot z\cdot\Delta x \le \cdot\frac{z}{z+\Delta x},$$

$$\cdot z + z\cdot\Delta x - \cdot z\cdot\Delta x + \cdot z\cdot\Delta x + \Delta x - \cdot z\cdot\Delta x = \cdot z + \Delta x - \cdot z\cdot\Delta x \le \cdot z,$$

$$\Delta x \le \cdot z.$$

Since we assumed $\Delta x \le 2\cdot z < 3\cdot z$ follows the inequality. Thus the inequality is valid for all positive z and Δx.

10 Sequences and Function values

In this section we develop sequences to compute function values, for example for *e*-function or trigonometric functions.

10.1 Sequences for the *e*-Function

To determine concrete function values, sequences are used, usually called Taylor sequences. The unknown function values – for example for the *e*-function – are approximated by a known function – for example a polynomial – that can be evaluated by arithmetic operators.

$$e^x = \sum_{k=}^{N} a_k \cdot x^k .$$

We have to determine the coefficients a_k. For this we assume that two equal functions must have the same derivative at each position. With this assumption we get the following equations, where we compute the k-th derivative $D^{(k)}(x)$ at the position $x = 0$ and $0^0 = 1$.

$$e^x =_{x=} \quad = \sum_{k=}^{N} a_k \cdot x^k =_{x=} \quad a \ ,$$

$$D_{e^x} =_{x=} \quad e^x =_{x=} \quad = D\left[\sum_{k=}^{N} a_k \cdot x^k \right] =_{x=} \quad \sum_{k=}^{N} k \cdot a_k \cdot x^{k-} \ =_{x=} \quad \cdot a \ = a \ ,$$

$$D_{e^x}^{(\ )} =_{x=} \quad e^x =_{x=} \quad = D^{(\ )}\left[\sum_{k=}^{N} a_k \cdot x^k \right] =_{x=} \quad \sum_{k=}^{N} k \cdot (k- \ ) \cdot a_k \cdot x^{k-} \ =_{x=} \quad \cdot a \ ,$$

$$D_{e^x}^{(\ )} =_{x=} \quad e^x =_{x=} \quad = D^{(\ )}\left[\sum_{k=}^{N} a_k \cdot x^k \right] =_{x=} \quad \sum_{k=}^{N} k \cdot (k- \ ) \cdot (k- \ ) \cdot a_k \cdot x^{k-} \ =_{x=} \quad \cdot \ \cdot a \ = \ a \ ,$$

$$D_{e^x}^{(m)} =_{x=} \quad e^x =_{x=} \quad = D^{(m)}\left[\sum_{k=}^{N} a_k \cdot x^k \right] =_{x=} \quad \sum_{k=}^{N} k \cdot \ldots \cdot (k-m+ \ ) \cdot a_k \cdot x^{k-m} =_{x=} \ m! \cdot a_m .$$

This is to be solved for a_k; thus we get the equation

$$e^x = \sum_{k=}^{N} \frac{x^k}{k!} .$$

We compute the value for $x = 1$, so we get $e^1 = e$, this is the value for the base of the *e*-Function; this is called *Euler number.* We get the following value for $N = 5$

$$e \ = \ + - + - + \frac{}{!} + \frac{}{!} + \frac{}{!} = \ + \ + \ , \ + \ , \qquad + \ , \qquad + \ , \qquad = \ , \qquad .$$

The exact value for the Euler number is 2,718281828... With six terms we are close to the correct value. With ten terms we get with 2,81828153 already five correct decimals, with 13 terms twelve, etc.

This method can be applied trustworthy for the *e*-Funktion. Two questions remain. At first we ask the error that is made if some terms are dropped. Then we will give some examples and show some problems with this method.

10.2 Residual term

What will be the error when we stop summation after N terms. To show this we develop the Taylor polynomial for the general case.

In general the following equation holds for each function f with derivative D_f, to be integrated in the interval $[u, x]$:

$$f(x)=f(u)+[I_{D_f}(x)-I_{D_f}(u)]=f(u)+I\left[D_f(t)\right]_{t=u}^{t=x}. \tag{39}$$

Here, u is called development position. At this position function values and values for all derivatives of the function f must be known, to compute the coefficients of the Taylor series.

From rules for partial integration follows for the definite integral

$$I\left[\frac{(x-t)^{n+}}{(n+\ )!}D_f^{(n+\ )}(t)\right]_{t=u}^{t=x}=I\left[\frac{(x-t)^{n}}{n!}D_f^{(n+\ )}(t)\right]_{t=u}^{t=x}+\left(\frac{(x-t)^{n+}}{(n+\ )!}D_f^{(n+\ )}(t)\right)_{t=u}^{t=x}=$$
$$=I\left[\frac{(x-t)^{n}}{n!}D_f^{(n+\ )}(t)\right]_{t=u}^{t=x}-\frac{(x-u)^{n+}}{(n+\ )!}D_f^{(n+\ )}(u).$$

Then follows after some rearrangements

$$I\left[\frac{(x-t)^{n}}{n!}D_f^{(n+\ )}(t)\right]_{t=u}^{t=x}=\frac{(x-u)^{n+}}{(n+\ )!}D_f^{(n+\ )}(u)+I\left[\frac{(x-t)^{n+}}{(n+\ )!}D_f^{(n+\ )}(t)\right]_{t=u}^{t=x}.$$

For $n = 0$ we get from equation (39) since

$$I_{D_f}(x)=\frac{x-u}{!}D_f(u)+I\left[\frac{x-t}{!}D_f^{(\ )}(t)\right]_{t=u}^{t=x}=(x-u)\cdot D_f(u)+I\left[(x-t)D_f^{(\ )}(t)\right]_{t=u}^{t=x}$$

the equation

$$f(x)=f(u)+(x-u)\cdot D_f(u)+I\left[(x-t)D_f^{(\ )}(t)\right]_{t=u}^{t=x}.$$

For $n = 1$ follows from this, since

$$I\left[\frac{(x-t)}{!}D_f^{(\ )}(t)\right]_{t=u}^{t=x}=I\left[(x-t)\cdot D_f^{(\ )}(t)\right]_{t=u}^{t=x}=\frac{(x-u)}{!}D_f^{(\ )}(u)+I\left[\frac{(x-t)}{!}D_f^{(\ )}(t)\right]_{t=u}^{t=x}.$$

the equation

$$f(x)=f(u)+(x-u)\cdot D_f(u)+\frac{(x-u)}{\ }D_f^{(\ )}(u)+I\left[\frac{(x-t)}{\ }D_f^{(\ )}(t)\right]_{t=u}^{t=x}.$$

Then we get for all n until N

$$f(x)=\sum_{k=}^{N}\frac{(x-u)^k}{k!}D_f^{(k)}(u)+I\left[\frac{(x-t)^N}{N!}D_f^{(N+\ )}(t)\right]_{t=u}^{t=x}. \quad \textit{Taylor polynomial} \tag{40}$$

This is called *Taylorpolynom* after *Brook Tayler* (the infinite series is called *Taylor series*). The last term is called residual term.

$$R_N(x)=I\left[\frac{(x-t)^N}{N!}D_f^{(N+\ )}(t)\right]_{t=u}^{t=x}. \quad \textit{Residual term of Taylor polynomial} \tag{41}$$

For our example of the e-Funktion is $x = 1$, $u = 0$, and thus the integral bounded by

$$\frac{\ }{N!}\cdot I\left[D_{\exp}^{(N+\ )}(t)\right]_{t=u}^{t=x}=\frac{\ }{N!}\cdot\left[D_{\exp}^{(N)}(t)\right]_{t=u}^{t=x}=\frac{\ }{N!}\cdot\left[D_{\exp}^{(N)}(x)-D_{\exp}^{(N)}(u)\right]=\frac{\ }{N!}\cdot\left[e^x-e^u\right]=\frac{\ }{N!}\cdot\left[e-\ \right]$$

10.3 Series for trigonometric functions

For trigonometric function we get in similar way for the Taylor polynomials developed at $u = 0$

$$\sin(x) = \sum_{k=\ ,\ ,\ ,\ldots}^{N} \pm\frac{(x-u)^k}{k!} \pm I\left[\frac{(x-t)^N}{N!}\cdot\cos t\right]_{t=}^{t=x} = \sum_{k=\ ,\ ,\ ,\ldots}^{N} \pm\frac{x^k}{k!} \pm I\left[\frac{(x-t)^N}{N!}\cdot\cos t\right]_{t=}^{t=x},$$

$$\cos(x) = \sum_{k=\ ,\ ,\ ,\ldots}^{N} \pm\frac{(x-u)^k}{k!} \mp I\left[\frac{(x-t)^N}{N!}\cdot\sin t\right]_{t=}^{t=x} = \sum_{k=\ ,\ ,\ ,\ldots}^{N} \pm\frac{x^k}{k!} \mp I\left[\frac{(x-t)^N}{N!}\cdot\sin t\right]_{t=}^{t=x}.$$

Both functions use only each second term of the Taylor series, since derivative of sine is sine again, and sin 0 = 0. Both series are alternating (changes sign). Explicitly written the formulae read

$$\sin(x) = x - \frac{x}{\ !} + \frac{x}{\ !} - \frac{x}{\ !} \pm\ldots\pm(-\ )^n\cdot\frac{x^{\mathrm{n}+}}{(\mathrm{n}+\ )!},$$

$$\cos(x) = \ -\frac{x}{\ !} + \frac{x}{\ !} - \frac{x}{\ !} \pm\ldots\pm(-\ )^n\cdot\frac{x^{\mathrm{n}}}{(\mathrm{n}\ )!},.$$

For cosine the first term must be 1 since cos 0 = 1. The residual terms are bounded since the absolute values of sine and cosine are bounded by 1. For small x the residual terms decrease with $1/N!$. That's why x must be choosen from the interval $[0, \pi/2]$.

Tangens is more complicated then sine and cosine, but can be computed from those functions, and for example from formula like

$$\tan x = \frac{\sin x}{\cos x} = \frac{\sqrt{\ -\cos\ x}}{\cos x} = \sqrt{\frac{\ }{\cos\ x} - \ }.$$

10.4 Logarithm

Logarithm is inverse of e-Funktion, the series of which we have already derived:

$$e^x = \sum_{k=}^{N} \frac{x^k}{k!} = \ + \frac{x}{\ } + \frac{x}{\ !} + \frac{x}{\ !} + \frac{x}{\ !} + \ldots + \frac{x^n}{n!}.$$

Since logarithm is not defined at $u = 0$ it will be derived at $u = 1$. For the derivatives we have $D_{\ln}(x) = 1/x$, $D_{\ln}^{(2)}(x) = -1/x^2$, $D_{\ln}^{(3)}(x) = 2/x^3$, $D_{\ln}^{(k)}(x) = (n-1)!/x^k$; then follows

$$\ln_{u=}\ x = \ln u + \sum_{k=}^{N} \frac{(x-u)^k}{k!} D_f^{(k)}(u) + I\left[\frac{(x-t)^N}{N!} D_f^{(N+\ )}(t)\right]_{t=u}^{t=x} =$$

$$= \sum_{k=}^{N} \frac{(x-\ )^k}{k}(-\ )^k + I\left[\frac{(x-t)^N}{N} t^{N+}\right]_{t=}^{t=x}.$$

This series

$$\ln_{u=}\ x = x - \ -\frac{(x-\ )}{\ } + \frac{(x-\ )}{\ } - \frac{(x-\ )}{\ } + \frac{(x-\ )}{\ } \pm\ldots$$

converges too slowly, since numerator increases only linearly. Thus we substitute $z = x-1$, this means $x = 1+z$, then follows from the last equation

$$\ln(\ +z) = z - \frac{z}{\ } + \frac{z}{\ } - \frac{z}{\ } + \frac{z}{\ } \pm\ldots$$

Now we substitute $z = 1-x$, this means $x = 1-z$, then follows from the same equation

$$\ln(\ -z) = -z - \frac{z}{\ } - \frac{z}{\ } - \frac{z}{\ } - \frac{z}{\ } - \ldots$$

Both series are are absolutely convergent for $|z| < 1$, then follows from subtraction the equation

$$\ln\frac{+z}{-z} = \ \cdot\left(z + \frac{z}{\ } + \frac{z}{\ } + \frac{z}{\ } + \frac{z}{\ } - \ldots \right)$$

Parameter $p = (1+z)/(1-z)$ of the logarithm yields because of $z = (p-1)/(p+1)$ for $p > 1$ always another positive value for z, where z is always less than 1. This series converges much better.

For example logarithm of 2 is ln 2 = 0,693147180560; then z = 0,693... and we add twelve terms to get twelve decimals accuracy. If p is too large, then z is too close to 1, and the series converges much worse. Given a number q we can divide by the biggest 2^k, thus $1 \leq r = q/2^k < 2$, and then we get for the logarithm

$$\ln q = \ln r \cdot 2^k = \ln r + \ln 2^k = \ln r + k \cdot \ln 2.$$

Now, only logarithms for numbers between 1 and 2 are to be computed. If $p < 1$, the formula $\ln p = -\ln(1/p)$ can be used.

This example shows that sometimes intricate considerations are required to apply Taylor series to practical problems.

11 Exercises

11.1 Integration of a constant

Show that the primitive of the function

$$f(x)=a$$

is

$$I_f(x)=a\cdot x.$$

Solution: Show the inequalities

$$a\cdot\Delta x\leq a\cdot(x+\Delta x)-a\cdot x=a\cdot\Delta x\leq a\cdot\Delta x.$$

From this follows the statement.

11.2 Integration of a step function

11.2.1 Integration of a simple step function

Let be $a(x) = 1$ if $x \geq a$ and 0 otherwise. The function

$$f(x) = 1 + a(x)$$

hat a step at the point $x = a$ from 1 to 2. Show that the primitive to this step function is

$$I_f(x) = x + x\cdot a(x) - a\cdot a(x).$$

Solution

This can be proved by the usual inequalities

$$\begin{aligned}
&(\ \ +a(x))\cdot\Delta x\leq I_f(x+\Delta x)-I_f(x)=\\
&\quad=x+\Delta x+(x+\Delta x)\cdot a(x+\Delta x)-a\cdot a(x+\Delta x)-x-x\cdot a(x)+a\cdot a(x)=\\
&\quad=\Delta x+(x+\Delta x-a)\cdot a(x+\Delta x)-(x-a)\cdot a(x)\leq\\
&\quad\leq(\ \ +a(x+\Delta x))\cdot\Delta x.
\end{aligned}$$

We distinguish three cases: $a(x) = a(x+\Delta x) = 0$; $a(x) =0$, $a(x+\Delta x) = 1$; $a(x) = a(x+\Delta x) = 1$. In the first case with $a(x) = a(x+\Delta x) = 0$ follows

$$(\ \ )\cdot\Delta x\leq\Delta x\leq(\ \ )\cdot\Delta x.$$

In the second case with $a(x) =0$, $a(x+\Delta x) = 1$ follows $x < a$ and $x+\Delta x \geq a$, thus

$$(\ \ )\cdot\Delta x\leq\Delta x+x+\Delta x-a\leq(\ \ +\ \ )\cdot\Delta x,$$

or

$$\leq x+\Delta x-a\leq\Delta x,$$

this follows from the preconditions. In the third case with $a(x) = a(x+\Delta x) = 1$ we get finally

$$\cdot\Delta x\leq\Delta x+(x+\Delta x-a)-(x-a)=\ \ \cdot\Delta x\leq\ \ \cdot\Delta x.$$

Thus inequalities (4) are also valid for non-continous functions f, where also in this case I_f is a continuos function.

11.2.2 Integration of a function with several steps

Let be $n(x)$ the integer part of x, that is $n(1,3)=1$, $n(7,25)=7$, $n(12,0)=12$, etc. Show that the primitive

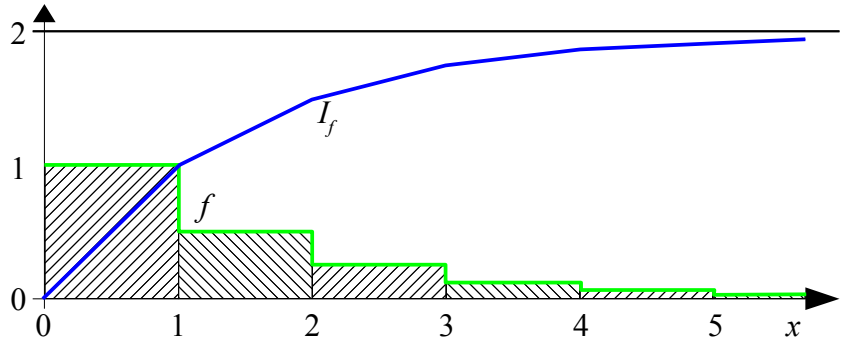
of

$$f(x)=(\ \ /\ \ )^{n(x)}$$

is

$$I_f(x)=\ \ +(x-n(x)-\ \ )\cdot(\ \ /\ \ )^{n(x)}.$$

The picture sketches the function f and its primitive I_f with unlimited number of steps.



Solution: Since the function f ascends monotonously, we have to prove the inequalities

$$\begin{aligned}(\ \ /\ \ )^{n(x)}\cdot\Delta x &\geq\ \ +(x+\Delta x-n(x+\Delta x)-\ \ )\cdot(\ \ /\ \ )^{n(x+\Delta x)}-\ \ -(x-n(x)-\ \ )\cdot(\ \ /\ \ )^{n(x)}=\\ &=(x+\Delta x-n(x+\Delta x)-\ \ )\cdot(\ \ /\ \ )^{n(x+\Delta x)}-(x-n(x)-\ \ )\cdot(\ \ /\ \ )^{n(x)}\geq\\ &\geq(\ \ /\ \ )^{n(x+\Delta x)}\cdot\Delta x.\end{aligned}$$

To show this consider at first the case that the interval $[x, x+\Delta x]$ lies within the decimal fraction of a number, so that $n(x+\Delta x) = n(x)$. Then follows

$$\begin{aligned}(\ \ /\ \ )^{n(x)}\cdot\Delta x &\geq(x+\Delta x-n(x+\Delta x)-\ \ )\cdot(\ \ /\ \ )^{n(x+\Delta x)}-(x-n(x)-\ \ )\cdot(\ \ /\ \ )^{n(x)}=\\ &=(x+\Delta x-n(x)-\ \ )\cdot(\ \ /\ \ )^{n(x)}-(x-n(x)-\ \ )\cdot(\ \ /\ \ )^{n(x)}=\\ &=\Delta x\cdot(\ \ /\ \ )^{n(x)}\geq\\ &\geq(\ \ /\ \ )^{n(x+\Delta x)}\cdot\Delta x=(\ \ /\ \ )^{n(x)}\cdot\Delta x.\end{aligned}$$

Since always hold equality, also the inequalities are correct. Now we consider the case that there is a step in the interval $[x, x+\Delta x]$ so that follows $n(x+\Delta x) = n(x)+1$. Then follows

$$\begin{aligned}(\ \ /\ \ )^{n(x)}\cdot\Delta x &\geq(x+\Delta x-n(x+\Delta x)-\ \ )\cdot(\ \ /\ \ )^{n(x+\Delta x)}-(x-n(x)-\ \ )\cdot(\ \ /\ \ )^{n(x)}=\\ &=(x+\Delta x-n(x)-\ \ -\ \ )\cdot(\ \ /\ \ )^{n(x)+}-(x-n(x)-\ \ )\cdot(\ \ /\ \ )^{n(x)}=\\ &=(-x+\Delta x+n(x)+\ \ )\cdot(\ \ /\ \ )^{n(x)+}\geq\\ &\geq(\ \ /\ \ )^{n(x)+}\cdot\Delta x.\end{aligned}$$

Reduce the positive factor $(\ \ /\ \ )^{n(x)+}$ and get

$$\cdot\Delta x\geq-x+\Delta x+n(x)+\ \ \geq\Delta x.$$

Subtraction of Δx yields

$$\Delta x\geq-x+n(x)+\ \ =\ \ -(x-n(x))\geq\ \ .$$

Since $x-n(x) < 1$ the right hand side holds. Since Δx ranges beyond a step, follows

$$\Delta x \geq 1-(x-n(x)),$$

so that also the left hand inequality is valid. Altogether the statement holds for all x and all $0 < \Delta x \leq 1$.

The figure 2 is the constant, so that $I_f(x)$ is the area in the interval $[0,x]$. For example follows

$$I_f(\ )=\ +(\ -n(\ )-\ )\cdot(\ /\ )^{n(\ )}=\ +(-\ )=\ ,$$

$$I_f(\ )=\ +(\ -n(\ )-\ )\cdot(\ /\ )^{n(\ )}=\ +(-\ )\cdot(\ /\ )=\ ,$$

$$I_f(\ ,\ )=\ +(\ ,\ -n(\ ,\ )-\ )\cdot(\ /\ )^{n(\ ,\ )}=\ +(\ ,\ -\ )\cdot(\ /\ )\ =\ -\frac{\ ,}{\ }=\ ,\qquad .$$

11.2.3 Integration of general jump function

Let be $n(x)$ as above. Show that the primitive of the function

$$f(x)=q^{n(x)}$$

with $q > 0$ is

$$I_f(x)=\frac{-q^{n(x)}}{-q}+(x-n(x))\cdot q^{n(x)}. \tag{42}$$

Solution: Show for $q < 1$

$$q^{n(x)}\cdot\Delta x\geq\frac{-q^{n(x+\Delta x)}}{-q}+(x+\Delta x-n(x+\Delta x))\cdot q^{n(x+\Delta x)}-\frac{-q^{n(x)}}{-q}-(x-n(x))\cdot q^{n(x)}\geq q^{n(x+\Delta x)}\cdot\Delta x.$$

For $q > 1$ the function is monotonously ascending, so that the inequality signs are to be reversed.

Let be $n(x+\Delta x) = n(x)$, then follows as in the last exercise

$$\begin{aligned}q^{n(x)}\cdot\Delta x&\geq\frac{-q^{n(x+\Delta x)}}{-q}+(x+\Delta x-n(x+\Delta x))\cdot q^{n(x+\Delta x)}-\frac{-q^{n(x)}}{-q}-(x-n(x))\cdot q^{n(x)}=\\&=\frac{-q^{n(x)}}{-q}+(x+\Delta x-n(x))\cdot q^{n(x)}-\frac{-q^{n(x)}}{-q}-(x-n(x))\cdot q^{n(x)}=\\&=\Delta x\cdot q^{n(x)}\geq q^{n(x+\Delta x)}\cdot\Delta x=q^{n(x)}\cdot\Delta x.\end{aligned}$$

Let be $n(x+\Delta x) = n(x)+1$, that means $\Delta x \geq 1 - (x - n(x)) > 0$; then follows as in the last exercise

$$\begin{aligned}q^{n(x)}\cdot\Delta x&\geq\frac{-q^{n(x+\Delta x)}}{-q}+(x+\Delta x-n(x+\Delta x))\cdot q^{n(x+\Delta x)}-\frac{-q^{n(x)}}{-q}-(x-n(x))\cdot q^{n(x)}=\\&=\frac{-q\cdot q^{n(x)}}{-q}+(x+\Delta x-n(x)-\ )\cdot q\cdot q^{n(x)}-\frac{-q^{n(x)}}{-q}-(x-n(x))\cdot q^{n(x)}=\\&=q^{n(x)}+((x+\Delta x-n(x)-\ )\cdot q-x+n(x))\cdot q^{n(x)}=\\&=((x+\Delta x-n(x)-\ )\cdot q-x+n(x)+\ )\cdot q^{n(x)}=\\&=((x-n(x)-\ )\cdot(q-\ )+q\cdot\Delta x)\cdot q^{n(x)}=\\&=((\ -(x-n(x)))\cdot(\ -q)+q\cdot\Delta x)\cdot q^{n(x)}\geq q^{n(x)+\ }\cdot\Delta x=q\cdot q^{n(x)}\cdot\Delta x.\end{aligned} \tag{43}$$

Divide by positive $q^{n(x)}$ simplifies these inequalities to

$$\Delta x\geq(\ -(x-n(x)))\cdot(\ -q)+q\cdot\Delta x\geq q\cdot\Delta x.$$

The left hand side holds because of $q < 1$ and $\Delta x \geq 1 - (x - n(x))$, so that

$$\Delta x\cdot(\ -q)\geq(\ -(x-n(x)))\cdot(\ -q).$$

The right hand inequality holds since $1 - (x - n(x)) > 0$ and thus

$$(\ -(x-n(x)))\cdot(\ -q)>\ .$$

For $q > 1$ we have to prove

$$q^{n(x)}\cdot\Delta x\leq\frac{-q^{n(x+\Delta x)}}{-q}+(x+\Delta x-n(x+\Delta x))\cdot q^{n(x+\Delta x)}-\frac{-q^{n(x)}}{-q}-(x-n(x))\cdot q^{n(x)}\leq q^{n(x+\Delta x)}\cdot\Delta x.$$

Analogous to (43)) follows

$$\Delta x\leq(\ -(x-n(x)))\cdot(\ -q)+q\cdot\Delta x\leq q\cdot\Delta x.$$

Here is with $\Delta x \geq 1 - (x - n(x))$ and because of $q > 1$, $1 - q < 0$ after multiplying $1 - q$:

$$\Delta x\cdot(\ -q)=\Delta x-\Delta x\cdot q\leq(\ -(x-n(x)))\cdot(\ -q).$$

Thus follows the left hand inequality. The second follows sind $(1 - (x - n(x)))(1 - q) < 0$.

The case $q = 1$ is trivial, since then $f(x) = 1$ is a constant, thus $I_f(x) = x$.

$$q^{n(x)}\cdot\Delta x=\Delta x\geq I_f(x+\Delta x)-I_f(x)=\Delta x\geq q^{n(x+\Delta x)}\cdot\Delta x=\Delta x.$$

Thus I_f is a solution to the problem of the summation of a geometric series, if x only takes integer numbers in (42); thus for $x = N$ follows

$$\sum_{i=}^{N} q^i=\frac{-q^N}{-q}.$$

11.3 Integrate the following functions

1. $x \cdot \cos x$
2. $x^2 \cdot \sin x$

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